TECHNISCHE UNIVERSITÄT DRESDEN

Fakultät Mathematik und Naturwissenschaften Fachrichtung Mathematik

Existence of Feller processes

Parametrix Construction

Feller processes behave locally like Lévy processes, but – in contrast to Lévy processes – they need not be homogeneous in time and space. The Lévy triplet (b, Q, ν) is replaced by an x-dependent triplet $(b(x), Q(x), \nu(x, dy))$. Typical examples are solutions to Lévy-driven SDEs, affine processes and stable-like processes. A (rich) Feller process can be characterized by its symbol

$$q(x,\xi) = -i\,b(x)\xi + \frac{1}{2}\xi Q(x)\xi + \int (1 - e^{iy\xi} + iy\xi \mathbb{1}_{|y|<1})\,\nu(x,dy), \qquad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$

Fundamental questions: When does a negative definite symbol $q(x,\cdot)$ give rise to a Feller process with symbol q? How to derive heat kernel estimates for the transition probability of the Feller process?

Example: NIG(-like) processes

Motivation

A one-dimensional Lévy process is Normal inverse Gaussian (NIG) if its characteristic exponent equals

$$\psi(\xi) = -ib\xi + \delta\sqrt{m^2 + (\xi - i\ell)^2} - \delta\sqrt{m^2 - \ell^2}.$$

Interesting for applications: state-space dependent parameters, i. e. $b = b(x)$, $\delta = \delta(x)$, ... ($\to$ NIG-like process)
Question: Under which assumptions on $b(\cdot)$, $\delta(\cdot)$, $m(\cdot)$, $\ell(\cdot)$ does there exist a Feller process with symbol

$$q(x,\xi) := -ib(x)\xi + \delta(x)\sqrt{m(x)^2 + (\xi - i\ell(x))^2} - \delta(x)\sqrt{m(x)^2 - \ell(x)^2}$$

Existence results for NIG-like processes

- So far: Existence results under very restrictive assumptions; require in particular **smoothness** of b, δ, m, ℓ ($\to$ Barndorff-Nielsen & Levendorskii)
- Now: Existence of NIG-like processes for **Hölder continuous** mappings b, δ, m, ℓ.
  Easy to prove using general existence result. Provides further additional information such as heat kernel estimates and richness of the domain of the generator (see below).

Existence result

Framework

$$q(x,\xi) = \psi_{\alpha(x)}(\xi)$$

for a family $(\psi_\alpha)_{\alpha\in I}$ of continuous negative definite functions and $I \subseteq \mathbb{R}^n$. Assumptions:

- $\psi_\alpha(\cdot)$ has a holomorphic extension to a certain domain $\Omega \subseteq \mathbb{C}$ for each $\alpha \in I$,
- $\partial_{\alpha_j}\psi_\alpha$ (exists and) extends holomorphically to Ω for all j
- ψ_α and $\partial_{\alpha_j}\psi_\alpha$ satisfy certain growth conditions.
- in dimension $d > 1$: rotational invariance of ψ_α

Parametrix construction

... gives existence of a Feller process $(X_t)_{t\geq 0}$ with symbol $q(x,\xi) := \psi_{\alpha(x)}(\xi)$ for **Hölder continuous** mappings $\alpha : \mathbb{R}^d \to I$. Additional information:

- $C_c^\infty(\mathbb{R}^d)$ is a core for the generator L,
- the $(L, C_c^\infty(\mathbb{R}^d))$-martingale problem is well-posed
- transition density p is fundamental solution to the Cauchy problem $\partial_t - L = 0$
- heat kernel estimates for p and its time derivative
- in dimension $d = 1$: irreducibility with respect to Lebesgue measure, heat kernel estimate for $\partial_x p(t, x, y)$

Applications

- Feller processes with symbols of varying order
- existence and uniqueness results for solutions of Lévy-driven SDEs
- variable order subordination, i. e. $q(x,\xi) = f_{\alpha(x)}(|\xi|^2)$ for a family $(f_\alpha)_{\alpha\in I}$ of Bernstein functions (e. g. relativistic stable-like, normal tempered stable-like, ...)

Reference Kühn, F.: Probability and Heat Kernel Estimates for Lévy(-Type) Processes. PhD Thesis, 2016.

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