Approximation of eigenfunctions of the fractional Laplacian

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Introduction

Method used for the estimation has been proposed by M. Kwaśnicki (2012). Main result obtained by Kwaśnicki provided to a construction of a matrix which eigenvectors could be considered as a discretization of original eigenfunctions. The method of construction the matrix V is given below.
Let $D \subseteq \mathbb{R}^d$ be an open set in $\mathbb{R}^d$, and let $\varepsilon > 0$. Let K_ε be the set of those $k \in \mathbb{Z}^d$ for which $D \cap \prod_{j=1}^{d} [k_j\varepsilon, (k_j+1)\varepsilon]$ is nonempty, and let $\kappa : \{1, 2, \dots, |K_\varepsilon|\} \to K_\varepsilon$ be the enumeration of elements of K_ε. Finally, let $\overline{v} = \sum_{k \in \mathbb{Z}^d} \|k\|^{-d-\alpha}$, where $\|k\| = \sqrt{\sum_{j=1}^{d} (|k_j| + 1)^2}$. Define a $|K_\varepsilon| \times |K_\varepsilon|$ matrix V with entries

$$V_{p,q} = -\frac{c_{d.\alpha}}{\varepsilon^\alpha} \|\kappa(p) - \kappa(q)\|^{-d-\alpha},$$

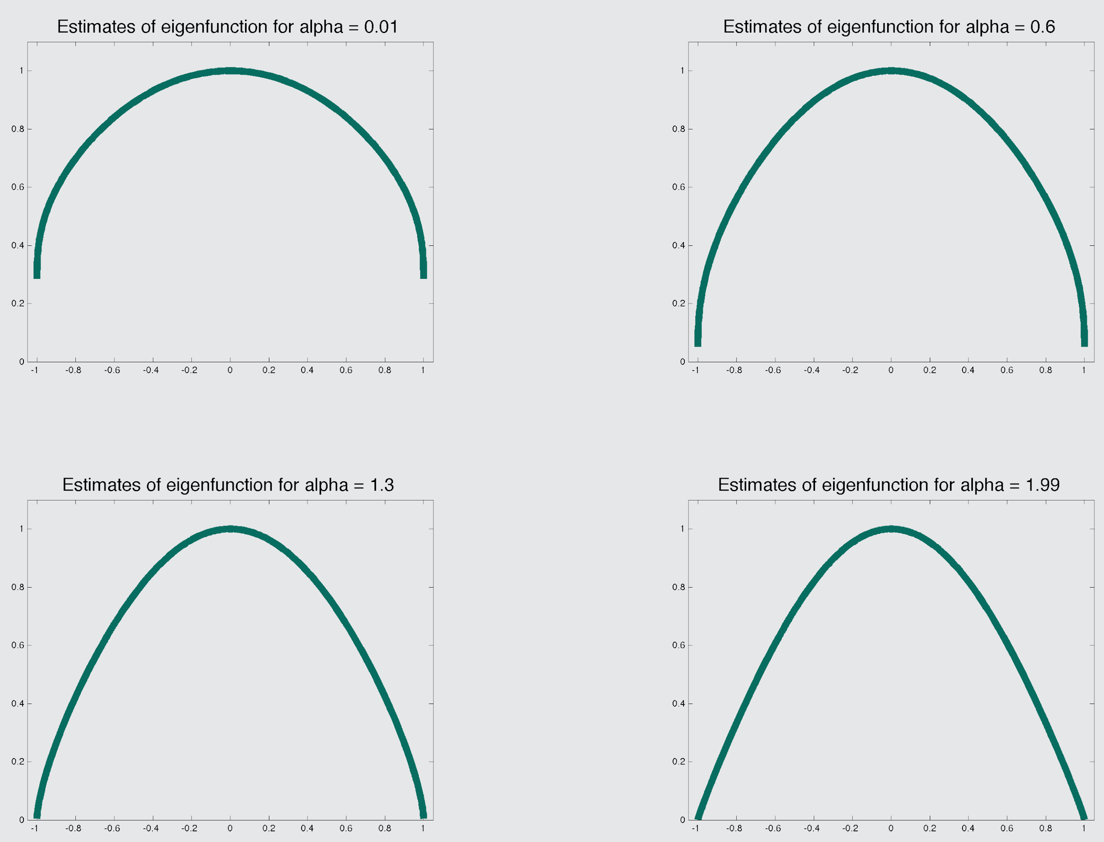
where $p, q = 1, 2, \dots, |K_\varepsilon|$, $p \neq q$;

$$V_{p,p} = -\frac{c_{d.\alpha}}{\varepsilon^\alpha} (\overline{v} - d^{-(d+\alpha)/2}),$$

for $p = 1, 2, \dots, |K_\varepsilon|$.
After few modification it has been used to obtain following results.

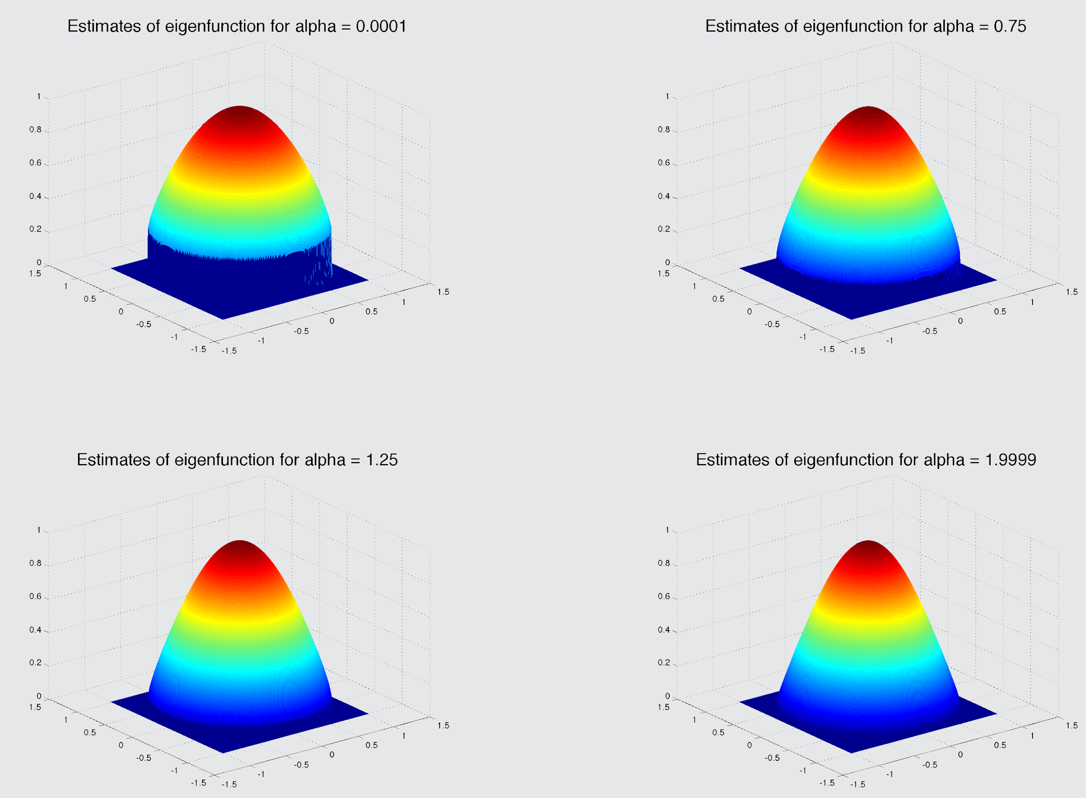
One-dimensional case

We divided an interval into 20001 parts so we have to consider matrices **20001 × 20001**.



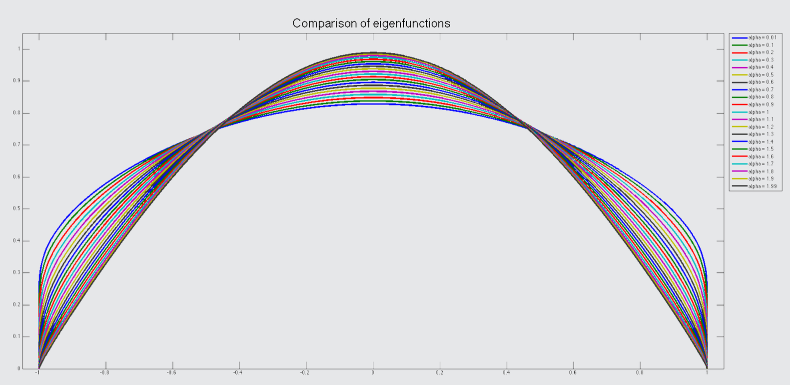
Two-dimensional case

We divided a disc into small squares in the way the diameter of ball contains 224 squares. If finally gives us under consideration a matrix of size **39669 × 39669**.



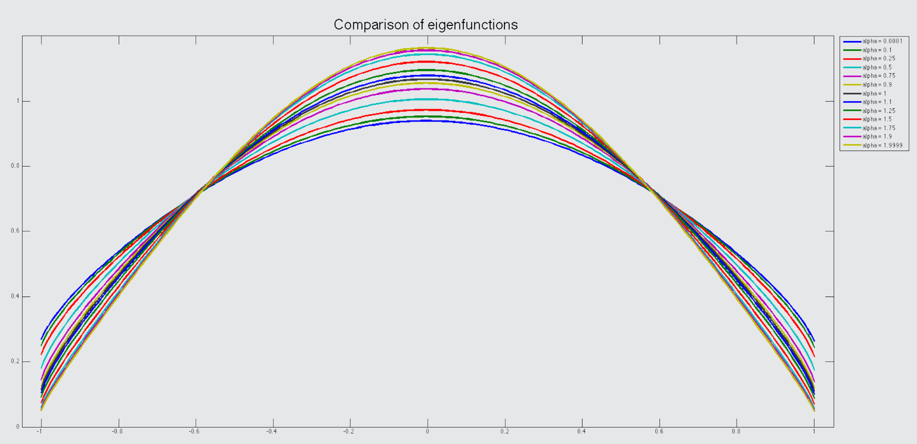
Convergence in one-dimensional case

Here we can see that the accuracy is not the only parameter affecting the convergence to a boundary limit but it directly depends also on α parameter.



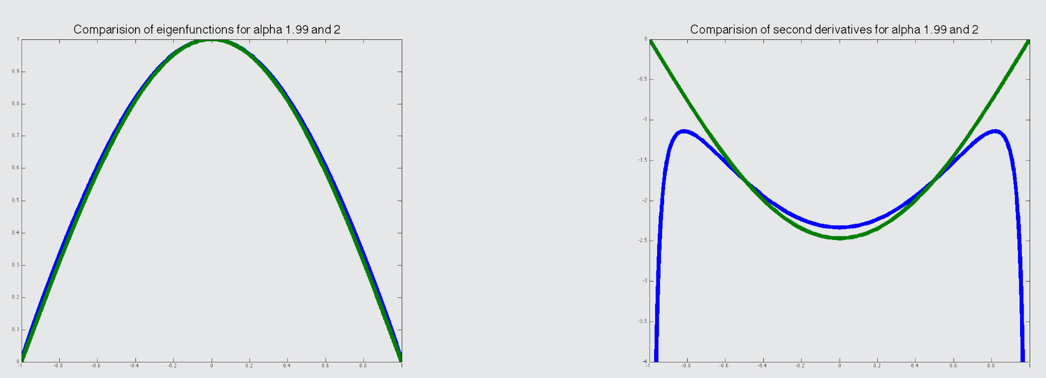
Convergence in two-dimensional case

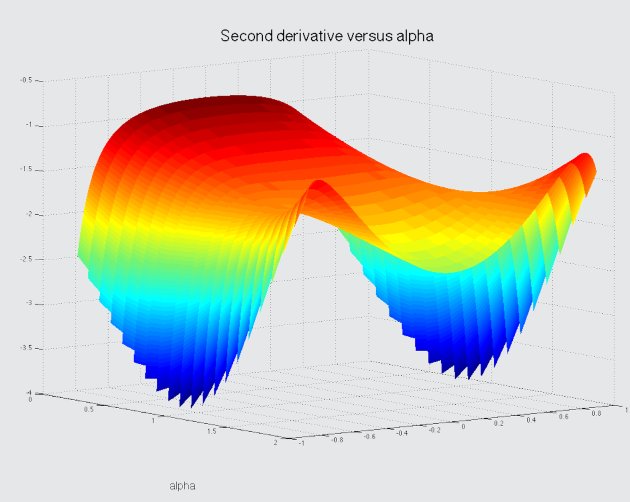
In two-dimensional case due to fewer linear division the convergence to boundary if significantly worse than in in one-dimensional case.



Second derivative in one-dimensional case

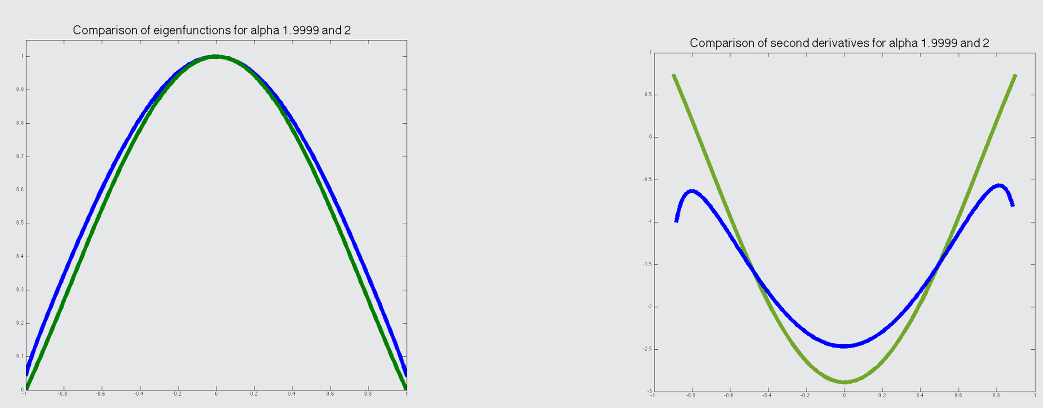
Here we can compare the second derivatives of our estimated eigfunction (for $\alpha = 1.99$) and proper eigenfunction for Laplacian $\mathbf{\Delta}$, equals $\cos(\frac{\pi}{2}x)$ whose second derivative is given by $-\left(\frac{\pi}{2}\right)^2 \cos(\frac{\pi}{2}x) < 0$.

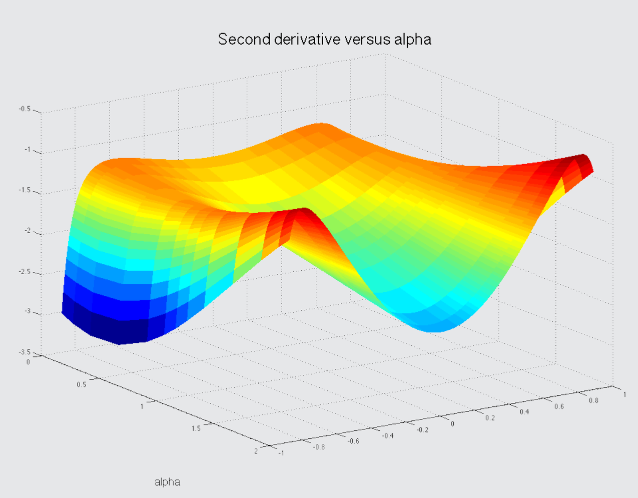




Second derivative in two-dimensional case

It is well known that eigenfunctions of $\mathbf{\Delta}$ are radially Bessel functions $J_n(k_{n,m}r)(\gamma \cos(n\theta) + \delta \sin(n\theta))$. Here we have estimates of eigenfunction for $\alpha = 1.9999$ and the exact eigenfunction for $\mathbf{\Delta}$.





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