Recurrent extensions of real self-similar Markov processes

Henry Pantí
UADY
henry.panti@correo.uady.mx

Abstract

In this paper we obtain necessary and sufficient conditions for the existence of recurrent extensions of real self-similar Markov processes. In doing so, we solve an old problem originally posed by Lamperti for positive self-similar Markov processes. We generalize Rivero's and Fitzsimmons results [3, 6, 7] to the real-valued case. Our main result ensures that a real self-similar Markov process with a finite hitting time of the point zero has a recurrent extension that leaves 0 continuously if and only if the MAP associated, via Lamperti transformation, satisfies the Cramér's condition.

Based on a joint work [5] with J.C. Pardo, CIMAT and Víctor Rivero, CIMAT.

1. Introduction

1.1 Real self-similar Markov processes

A *real self-similar Markov process* (rssMp for short) with *self-similarity index* $\alpha > 0$ is a standard Markov process $X = (X_t)_{t\geqslant 0}$ with probability laws $\mathrm{P} = (\mathrm{P}_x)_{x\in\mathbb{R}}$ which satisfies the following *scaling property*: for all $c > 0$,

$$\{(cX_{tc^{-\alpha}})_{t\geqslant 0}, \mathrm{P}_x\} \overset{\text{Law}}{=} \{(X_t)_{t\geqslant 0}, \mathrm{P}_{cx}\}, \quad \forall\, x \in \mathbb{R}.$$

Let T_0 be the first hitting time of zero for X, i.e.,

$$T_0 = \inf\{t > 0 : X_t = 0\}.$$

Assumption. The state 0 is recurrent, regular for itself and will be considered as a cemetery point.

A standard process $\tilde{X}$ is called a *recurrent extension* of (X, P) if $\tilde{X}$ behaves as (X, P) until T_0 and the state zero is a regular and recurrent state for $(\tilde{X}, \mathrm{P})$.

1.2 Markov additive processes

Let E be a finite state space and $(\mathcal{G}_t)_{t\geqslant 0}$ a standard filtration. A càdlàg process (ξ, J) in $\mathbb{R} \times E$ with law $\mathbb{P}$ is called *Markov additive process* (MAP for short) with respect to $(\mathcal{G}_t)_{t\geqslant 0}$ if $(J(t))_{t\geqslant 0}$ is a continuous time Markov chain in E, and the following property is satisfied, for any $i \in E$, $s, t \geqslant 0$:

given $\{J(t) = i\}$, the pair $(\xi(t+s) - \xi(t), J(t+s))$ is independent of $\mathcal{G}_t$, and has the same distribution as $(\xi(s) - \xi(0), J(s))$ given $\{J(0) = i\}$.

We use the following notation:

$$\mathbb{P}_{z,i}(\cdot) = \mathbb{P}(\cdot|\xi(0) = z, J(0) = i), \quad z \in \mathbb{R}, i \in E.$$

With the convention: $\mathbb{P}_i = \mathbb{P}_{0,i}$.

A characterization of a MAP is as follows: The pair (ξ, J) is a MAP if and only if, for each $i, j \in E$, there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geqslant 0}$, an a sequence of iid random variables $(U_{i,j}^n)_{n\geqslant 0}$ independent of the chain J, such that $\sigma_0 = 0$ and $(\sigma_n)_{n\geqslant 0}$ are the jump times of J, and the process ξ has the following representation:

$$\xi(t) = \mathbf{1}_{\{n>0\}}(\xi_{\sigma_n-} + U^n_{J(\sigma_n-),J(\sigma_n)}) + \xi^n_{J(\sigma_n)}(t - \sigma_n),$$

for $t \in [\sigma_n, \sigma_{n+1})$, $n \geqslant 0$.

Assumption. J is an irreducible Markov chain with equilibrium distribution π.

Let $Q = (q_{i,j})_{i,j\in E}$ be the rate matrix of the chain J. For each $i \in E$, let ψ_i be the Laplace exponent of the Lévy process ξ_i. Let G be the matrix with entries $G_{ij}(z) = \mathbb{E}[e^{zU_{ij}}]$ (with the convention that $U_{ij} = 0$ if $q_{ij} = 0$, $i \neq j$, and also set $U_{ii} = 0$ for each $i \in E$). Then, the matrix-valued function F given by

$$F(z) = \mathrm{diag}(\psi_1(z), \ldots, \psi_N(z)) + Q \circ G(z),$$

for all $z \in \mathbb{C}$ where the elements on the right are defined and $\circ$ indicates the Hadamard multiplication, satisfies

$$\mathbb{E}_{0,i}[e^{z\xi(t)}; J(t) = j] = \left(e^{F(z)t}\right)_{ij}, \quad i, j \in E,$$

for all $z \in \mathbb{C}$ where one side of the equality is defined. The matrix F is called the *matrix exponent* of the MAP ξ.

The matrix $F(z)$ has a real simple eigenvalue $\kappa(z)$, which is smooth and convex on its domain and larger than the real part of all its other eigenvalues. Furthermore, the corresponding right-eigenvector $v(z)$ may be chosen so that $v_i(z) > 0$ for every $i \in E$, and normalised as $\pi v(z) = 1$. This allow us to construct the Wald martingale:

$$M(t, \gamma) = e^{\gamma\xi(t) - \kappa(\gamma)t}\frac{v_{J(t)}(\gamma)}{v_{J(0)}(\gamma)}, \quad t \geqslant 0,$$

for some γ such that the right-hand side is defined. $M(\cdot, \gamma)$ is a unit-mean martingale with respect to $\mathcal{G}_t$ under any initial distribution of $(\xi(0), J(0))$. Thus, we can define the change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}^{(\gamma)}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{G}_t} = M(t, \gamma).$$

Under $\mathbb{P}^{(\gamma)}$, ξ is still a MAP with matrix exponent $F^{(\gamma)}$:

$$F^{(\gamma)}(z) = (\mathrm{diag}(v_i(\gamma)))^{-1}[F(z+\gamma) - \kappa(\gamma)\mathrm{Id}]\,\mathrm{diag}(v_i(\gamma)).$$

Given the MAP ξ with probabilities $\mathbb{P}_{z,i}$, the dual process of ξ is a MAP with probabilities $\hat{\mathbb{P}}_{z,i}$ and with matrix exponent:

$$\hat{F}(z) = \mathrm{diag}(\psi_1(-z), \ldots, \psi_N(-z)) + \hat{Q} \circ G(-z)^T,$$

where $\hat{Q}$ has entries given by $\hat{q}_{i,j} = \pi_j q_{j,i}(\pi_i)^{-1}$, $i, j \in E$.

1.3 Lamperti transformation

Recently in [1] it is established that for any rssMp there is a MAP (ξ, J) in $\mathbb{R} \times \{-1, 1\}$ such that under P_x, $x \neq 0$, the process X can be represented as

$$X_t = \exp\left\{\xi(\tau(t))\right\}J(\tau(t)), \qquad t \geqslant 0,$$

where

$$\tau(t) = \inf\left\{s \geqslant 0 : \int_0^s \exp\left\{\alpha\xi(u)\right\}\mathrm{d}u \geqslant t\right\},$$

and $(\xi(0), J(0)) = (\log x, [x])$, with $[x]$ the sign function.

1.4 Excursion measure

Let $\mathbb{D}$ be the space of càdlàg paths defined on $[0, \infty)$ with values in $\mathbb{R}$, and endowed with the Skorohod topology. Let $(\mathcal{F}_t)_{t\geqslant 0}$ be the natural filtration generated by the canonical process X.

We say that a σ-finite measure on $(\mathbb{D}, \mathcal{F}_\infty)$ having infinite mass is an excursion measure compatible with (X, P) if the following are satisfied:

1. $\mathbf{n}$ is carried by
$$\{\omega \in \mathbb{D} : T_0(\omega) > 0, X_t(\omega) = 0, \forall\, t \geqslant T_0\};$$
2. for every bounded $\mathcal{F}_\infty$-measurable H and each $t > 0$, $\Lambda \in \mathcal{F}_t$
$$\mathbf{n}(H \circ \theta_t, \Lambda \cap \{t < T_0\}) = \mathbf{n}(\mathrm{E}_{X_t}(H), \Lambda \cap \{t < T_0\}),$$
where θ_t denotes the shift operator;
3. $\mathbf{n}(1 - e^{T_0}) < \infty$.

We say that $\mathbf{n}$ is self-similar if it has the following scaling property: there exists a $\gamma \in (0, 1)$ such that for all $a > 0$, holds

$$H_a\mathbf{n} = a^{\gamma\alpha}\mathbf{n},$$

where the measure $H_a\mathbf{n}$ is the image of $\mathbf{n}$ under the mapping $H_a : \mathbb{D} \to \mathbb{D}$, defined by $H_a(\omega)(t) = a\omega(a^{-\alpha}t)$, $t \geqslant 0$. The parameter γ is called the *index of self-similarity* of $\mathbf{n}$.

2. Main results

Cramér's condition. There exists $z_0 > 0$, such that $F(z)$ is well defined on $(0, z_0)$ and there exists $\theta \in (0, z_0)$, such that $\kappa(\theta) = 0$. The value θ is called the *Cramér number*.

Set $\mathbb{P}^\sharp := \mathbb{P}^{(\theta)}$, with θ satisfying the Cramér's condition. Denote by $\hat{\mathbb{P}}^\sharp$ its dual. Let I be the functional exponential of the MAP ξ:

$$I = \int_0^\infty \exp\{\alpha\xi(t)\}\mathrm{dt}.$$

It can be shown that any rssMp for which 0 is a regular and recurrent state either leaves 0 continuously or by a jump.

Theorem 1. *Let (X, P) be a rssMP with index $\alpha > 0$, which hits its cemetery point 0 in a finite time P-$a.s.$ Let $((\xi, J), \mathbb{P})$ be the MAP associated with (X, P) via the Lamperti's transformation. Then the following conditions are equivalent:*

(i) there exist a Cramér number $\theta \in (0, \alpha)$;

(ii) there exist a recurrent extension of (X, P) that leaves 0 continuously and such that its associated excursion measure from 0, $\mathbf{n}$, satisfies $\mathbf{n}(1 - e^{-T_0}) = 1$.

In this case, the recurrent extension in (ii) is unique and the entrance law associated with the excursion measure $\mathbf{n}$ is, for any f bounded measurable, given by

$$\mathbf{n}\left(f(X_t), t < T_0\right) = \frac{1}{C_{\alpha,\theta}t^{\theta/\alpha}}\left[\frac{\pi_1}{v_1}\hat{\mathbb{E}}_1^\sharp\left(f\left(\frac{t^{1/\alpha}}{I^{1/\alpha}}\right)I^{\theta/\alpha-1}\right) + \frac{\pi_{-1}}{v_{-1}}\hat{\mathbb{E}}_{-1}^\sharp\left(f\left(-\frac{t^{1/\alpha}}{I^{1/\alpha}}\right)I^{\theta/\alpha-1}\right)\right],$$

where θ satisfies the condition (i) and

$$C_{\alpha,\theta} = \Gamma(1 - \theta/\alpha)\left[\frac{\pi_1}{v_1}\hat{\mathbb{E}}_1^\sharp\left(I^{\theta/\alpha-1}\right) + \frac{\pi_{-1}}{v_{-1}}\hat{\mathbb{E}}_{-1}^\sharp\left(I^{\theta/\alpha-1}\right)\right].$$

Furthermore, $\mathbf{n}$ is self-similar with index $\gamma = \theta/\alpha$.

Theorem 2. *Let $\beta \in (0, \alpha)$. The following are equivalent:*

(i) $\kappa(\beta) < 0$.

(ii) $\mathbb{E}_i(I^{\beta/\alpha}) < \infty$, for $i = -1, 1$.

(iii) The process (X, T_0) admits an extension $\tilde{X}$, that is a self-similar recurrent Markov process, and leaves 0 by a jump and whose associated excursion measure $\mathbf{n}^\beta$ is such that

$$\mathbf{n}^\beta(X_{0+} \in \mathrm{d}x) = b_{\alpha,\beta}^{[x]}|x|^{(\beta+1)}\mathrm{d}x$$

where $b_{\alpha,\beta}^1, b_{\alpha,\beta}^{-1}$ satisfy

$$b_{\alpha,\beta}^1\mathbb{E}_1(I^{\beta/\alpha}) + b_{\alpha,\beta}^{-1}\mathbb{E}_{-1}(I^{\beta/\alpha}) = \frac{\beta}{\Gamma(1 - \beta/\alpha)}.$$

If one of these conditions hold then $\mathbf{n}^\beta$ is self-similar with index $\gamma = \beta/\alpha$.

3. Examples

Example 1. Let (X, P) be an α-stable process, $\alpha \in (1, 2)$. The matrix exponent of ξ, the MAP associated with (X, P) via Lamperti transformation, is

$$F(z) = \begin{pmatrix} -\dfrac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho} - z)\Gamma(1 - \alpha\hat{\rho} + z)} & \dfrac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \\ \dfrac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} & -\dfrac{\Gamma(\alpha - z)\Gamma(1 + z)}{\Gamma(\alpha\rho - z)\Gamma(1 - \alpha\rho + z)} \end{pmatrix},$$

for $\mathrm{Re}(z) \in (-1, \alpha)$. Here the Cramér number is $\theta = \alpha - 1$.

Example 2. Let (X, P) be an α-stable process, $\alpha \in (0, 1)$. Let $((\xi, J), \mathbb{P})$ be the MAP associated to (X, P) via the Lamperti transformation. It is well known that for $\alpha \in (0, 1)$, (X, P) never reaches the point zero, then (ξ, J) drifts towards to $+\infty$ (see [2]). We consider the dual of ξ, $((\xi, J), \hat{\mathbb{P}})$, which drifts to $-\infty$. Thus, the rssMp associated with $((\xi, J), \hat{\mathbb{P}})$, via the Lamperti transformation, reaches the point zero at finite time. The matrix exponent of $((\xi, J), \hat{\mathbb{P}})$ is given by

$$\hat{F}(z) = \begin{pmatrix} -\dfrac{\Gamma(\alpha + z)\Gamma(1 - z)}{\Gamma(\alpha\hat{\rho} + z)\Gamma(1 - \alpha\hat{\rho} - z)} & \dfrac{\Gamma(\alpha + z)\Gamma(1 - z)}{\Gamma(\alpha\hat{\rho})\Gamma(1 - \alpha\hat{\rho})} \\ \dfrac{\Gamma(\alpha + z)\Gamma(1 - z)}{\Gamma(\alpha\rho)\Gamma(1 - \alpha\rho)} & -\dfrac{\Gamma(\alpha + z)\Gamma(1 - z)}{\Gamma(\alpha\rho + z)\Gamma(1 - \alpha\rho - z)} \end{pmatrix},$$

for $\mathrm{Re}(z) \in (-\alpha, 1)$ and the Cramér number is $\theta = 1 - \alpha$. Observe that condition in Theorem 1 (i) holds if and only if $\alpha > 1/2$. Thus, the recurrent extension that leaves 0 continuously exists whenever $\alpha > 1/2$.

The aforementioned process can be considered as the α-stable process conditioned to be continuously absorbed at the origin (see [4] for more details).

References

[1] L. Chaumont, H. Pantí, and V. Rivero. The Lamperti representation of real-valued self-similar Markov processes. *Bernoulli*, 19(5B):2494–2523, 2013.

[2] S. Dereich, L. Doering, and A. E. Kyprianou. Real Self-Similar Processes Started from the Origin. *ArXiv 1501.00647*, 2015.

[3] P. Fitzsimmons. On the existence of recurrent extensions of self-similar markov processes. *Electron. Commun. Probab.*, 11:230–241, 2006.

[4] A. E. Kyprianou, V. M. Rivero, and W. Satitkanitkul. Conditioned real self-similar Markov processes. *ArXiv e-prints*, Oct. 2015.

[5] H. Pantí, J. Pardo, and V. Rivero. Recurrent extensión of real self-similar markov processes. Preprint, 2016.

[6] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. *Bernoulli*, 11(3):471–509, 2005.

[7] V. Rivero. Recurrent extensions of self-similar Markov processes and Cramér's condition. II. *Bernoulli*, 13(4):1053–1070, 2007.

8th International Conference on Lévy Processes, Angers, France, June 25-29, 2016.