Generalization of refracted Lévy processes and its application to exit problem

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1. Abstract

Generalizing Kyprianou–Loeffen's refracted Lévy processes, we define a new refracted Lévy process which is a Markov process whose behaviors during the period of non-negative values and during that of negative ones are Lévy processes different from each other. To construct it we utilize the excursion theory. We study its exit problem and the potential measures of the killed processes. This study is the joint work with Professor Kouji Yano (Kyoto University).

2. Preliminaries

Let $Z = \{Z_t : t \geq 0\}$ be a spectrally negative Lévy process such that $-Z$ is not a subordinator and $\mathbb{P}^Z_x$ is the law of Z when issued from x .
Define its Laplace exponent as

$$\Psi_Z(q) := \log \mathbb{E}^Z_0(e^{qZ_1}) \tag{1}$$

for each $q \geq 0$ and the right inverse as

$$\Phi_Z(\theta) = \inf\{q \geq 0 : \Psi(q) = \theta\} \tag{2}$$

for each $\theta \geq 0$.
When Z has paths of bounded variation we may always write

$$\Psi_Z(q) = \delta_Z q - \int_{(-\infty,0)} (1 - e^{qx})\Pi_Z(dx) \tag{3}$$

where $\delta_Z > 0$ and Π_Z is a measure concentrated on $(-\infty, 0)$ such that $\int_{(-\infty,0)}(1 \wedge |x|)\Pi_Z(dx) < \infty$.
When Z has paths of unbounded variation we may always write

$$\Psi_Z(q) = \gamma_Z q + \frac{1}{2}\sigma_Z^2 q^2 - \int_{(-\infty,0)} \left(1 - e^{qy} + qy1_{(-1,0)}\right) \Pi_Z(dy) \tag{4}$$

where $\gamma_Z \in \mathbb{R}$, $\sigma_Z \geq 0$ and Π_Z is a measure concentrated on $(-\infty, 0)$ such that $\int_{(-\infty,0)}(1 \wedge x^2)\Pi_Z(dx) < \infty$.
For all stochastic process Z, $q > 0$, $t \geq 0$, $x \in \mathbb{R}$ and positive or bounded measurable function f, define $R^{(q)}_Z f(x) := \mathbb{E}^Z_x \left(\int_0^\infty e^{-qs} f(Z_s)ds\right)$ and $P^Z_t f(x) := \mathbb{E}^Z_x (f(Z_t))$.

Definition (See, e.g., Kyprianou(2014))

For each $q \geq 0$, we define $W^{(q)}_Z : \mathbb{R} \to [0, \infty)$ such that $W^{(q)}_Z = 0$ on $(-\infty, 0)$ and $W^{(q)}_Z$ on $[0, \infty)$ is continuous satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}_Z(x)dx = \frac{1}{\Psi_Z(\beta) - q} \tag{5}$$

for all $\beta > \Phi_Z(q)$. This function $W^{(q)}_Z$ is called the q-scale function of Z.

Define

$$\tau^+_x := \inf\{t > 0 : Z_t > x\} \text{ and } \tau^-_x := \inf\{t > 0 : Z_t < x\}, \tag{6}$$

for all $x \in \mathbb{R}$.

Let $b < 0 < a$ be constants.

3. Kyprianou–Loeffen's Refracted Lévy processes

Let X and Y be spectrally negative Lévy processes satisfying
$Y_t = X_t + \alpha t$ $(\alpha > 0)$.

Definition (Kyprianou–Loeffen(2010) translated)

U is called a refracted Lévy process when U is a solution of the stochastic differential equation

$$U_t - U_0 = Y_t - \alpha \int_0^t 1_{\{U_s > 0\}} ds, \quad t \geq 0. \tag{7}$$

Theorem (Kyprianou–Loeffen(2010))

The stochastic differential equation (7) has a unique pathwise solution.

For all $x, y \in \mathbb{R}$ and $q \geq 0$, define

$$w^{(q)}_U(x; y) := W^{(q)}_Y(x - y) + \alpha 1_{(x \geq 0)} \int_b^x W^{(q)}_X(x - z) W^{(q)\prime}_Y(z - y)dz. \tag{8}$$

Theorem (Kyprianou–Loeffen(2010))

For all $x \in (b, a)$, we have

$$\mathbb{E}^U_x \left(e^{-q\tau^+_a} : \tau^+_a < \tau^-_b\right) = \frac{w^{(q)}_U(x; b)}{w^{(q)}_U(a; b)}. \tag{9}$$

Theorem (Kyprianou–Loeffen(2010))

For all $q > 0$ and non-negative function f, we have

$$\underline{\overline{R}}^{(q)}_U f(x) = \mathbb{E}^U_x \left(\int_0^{\tau^-_b \wedge \tau^+_a} e^{-qt} f(U_t)dt\right) = \int_b^a f(y)\underline{\overline{r}}^{(q)}_U(x, y)dy, \tag{10}$$

$$\underline{\overline{r}}^{(q)}_U(x, y) = \begin{cases} \frac{w^{(q)}_U(x;b)}{w^{(q)}_U(a;b)} w^{(q)}_U(a; y) - w^{(q)}_U(x; y) & x \in [b, 0] \\ \frac{w^{(q)}_U(x;b)}{w^{(q)}_U(a;b)} W^{(q)}_X(a - y) - W^{(q)}_X(x - y) & x \in (0, a]. \end{cases} \tag{11}$$

4. Main results

We assume that X and Y are spectrally negative Lévy processes.
When X has bounded variation paths, we define a refracted Lévy process U as a unique strong solution of the stochastic differential equation

$$U_t = U_0 + \int_{(0,t]} 1_{\{U_{s-} \geq 0\}} dX_s + \int_{(0,t]} 1_{\{U_{s-} < 0\}} dY_s. \tag{12}$$

When X has unbounded variation paths and no Gaussian part, we define a refracted Lévy process as follows:
Let n^X denote an excursion measure of X. Define the law of the stopped process $\mathbb{P}^{U^0}_x$ $(x \neq 0)$ and the excursion measure n^U by

$$\mathbb{P}^{U^0}_x \left(F\left((U_t)_{t<\tau^-_0}, (U_{t+\tau^-_0})_{t\geq 0}\right)\right) = \mathbb{P}^X_x \left(\mathbb{E}^{Y^0}_x \left(F\left(w, (Y^0_t)_{t\geq 0}\right)\right) \Big|_{\substack{x = X(\tau^-_0) \\ w = (X_t : t < \tau^-_0)}}\right) \tag{13}$$

$$n^U \left(F\left((U_t)_{t<\tau^-_0}, (U_{t+\tau^-_0})_{t\geq 0}\right)\right) = n^X \left(\mathbb{E}^{Y^0}_x \left(F\left(w, (Y^0_t)_{t\geq 0}\right)\right) \Big|_{\substack{x = X(\tau^-_0) \\ w = (X_t : t < \tau^-_0)}}\right) \tag{14}$$

where Y^0 is a stopped process of Y at 0. Using the excursion theory, we can construct a strong Markov process U without stagnancy at 0. We call this U a refracted Lévy process.
For all $y < 0$, $x \in \mathbb{R}$ and $q \geq 0$, define

$$w^{(q)}_U(x; y) = \begin{cases} W^{(q)}_X(x) W^{(q)}_Y(-y) \left(\Psi'_X(0) \vee 0\right) \\ \quad + \int_0^\infty dv \int_{(-\infty,0)} \left(W^{(q)}_X(x) W^{(q)}_Y(-y) e^{\Phi_Y(0)u} \right. \\ \quad \left. - W^{(q)}_Y(u - y) W^{(q)}_X(x - v)\right) \Pi_X(du - v), & x \in (0, \infty) \\ W^{(q)}_Y(x - y), & x \in [b, 0]. \end{cases} \tag{15}$$

Theorem (N–Yano)

For all $x \in (b, a)$, we have

$$\mathbb{E}^U_x \left(e^{-q\tau^+_a} : \tau^+_a < \tau^-_b\right) = \frac{w^{(q)}_U(x; b)}{w^{(q)}_U(a; b)}. \tag{16}$$

Theorem (N–Yano)

For all $q > 0$ and non-negative function f, we have

$$\underline{\overline{R}}^{(q)}_U f(x) = \mathbb{E}^U_x \left(\int_0^{\tau^-_b \wedge \tau^+_a} e^{-qt} f(U_t)dt\right) = \int_b^a f(y)\underline{\overline{r}}^{(q)}_U(x, y)dy, \tag{17}$$

$$\underline{\overline{r}}^{(q)}_U(x, y) = \begin{cases} \frac{w^{(q)}_U(x;b)}{w^{(q)}_U(a;b)} w^{(q)}_U(a; y) - w^{(q)}_U(x; y) & x \in [b, 0] \\ \frac{w^{(q)}_U(x;b)}{w^{(q)}_U(a;b)} W^{(q)}_X(a - y) - W^{(q)}_X(x - y) & x \in (0, a]. \end{cases} \tag{18}$$

Theorem (N–Yano)

Every our refracted Lévy process is a Feller process.

We approximate U of unbounded variation by a sequence $\{U^{(n)}\}_{n\in\mathbb{N}}$ of bounded variation which is obtained from U by removing small jumps. We prove $\{U^{(n)}\}_{n\in\mathbb{N}}$ converges to U in resolvent and distribution.
Let Z be a spectrally negative Lévy process. Let Ψ_Z denote the Laplace exponent represented by (4). For $n \in \mathbb{N}$, we define

$$\Psi_{Z^{(n)}}(q) = \gamma_Z q - \sigma_Z^2 n^2 \left(1 - e^{-\frac{1}{n}q} - q\frac{1}{n}\right) - \int_{(-\infty,-\frac{1}{n})} \left(1 - e^{qy} + qy1_{(-1,-\frac{1}{n})}\right) \Pi_Z(dy). \tag{19}$$

If we denote by $Z^{(n)}$ a Lévy process with Laplace exponent $\Psi_{Z^{(n)}}$, it is actually a compound Poisson process with positive drift.
Let $\{X^{(n)}\}_{n\in\mathbb{N}}$ and $\{Y^{(n)}\}_{n\in\mathbb{N}}$ be sequences of compound Poisson processes constructed from X and Y by the above technique. Let $U^{(n)}$ be a unique strong solution of (12) constructed by $X^{(n)}$ and $Y^{(n)}$.

Theorem (N–Yano)

For all $q > 0$ and continuous function f satisfying $\lim_{x\uparrow\infty} f(x) = \lim_{x\downarrow-\infty} f(x) = 0$, we have

$$R^{(q)}_{U^{(n)}} f(x) \to R^{(q)}_U f(x) \text{ uniformly as } n \uparrow \infty, \tag{20}$$

$$P^{U^{(n)}}_t f(x) \to P^U_t f(x) \text{ uniformly as } n \uparrow \infty. \tag{21}$$

Consequently, $\{(U^{(n)}, \mathbb{P}^{U^{(n)}}_x)\}_{n\in\mathbb{N}}$ converges in distribution to U under $\mathbb{P}^U_x$ for all $x \in \mathbb{R}$.

References

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