Estimates of Dirichlet heat kernel for symmetric Markov processes

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1 Setup

Consider symmetric Markov processes whose jumping kernels J, (1) satisfying weak scaling condition and (2) decaying exponential with damping exponent $\beta \in [0, \infty]$.

For $0 < \underline{\alpha} \le \overline{\alpha} < 2$, let $\phi \in C^1(0,\infty)$ be an increasing function on $[0,\infty)$ satisfying that there exist positive constants $\underline{c} \le 1$ and $1 \le \overline{C}$ such that

$$\underline{c}\left(\frac{R}{r}\right)^{\underline{\alpha}} \le \frac{\phi(R)}{\phi(r)} \le \overline{C}\left(\frac{R}{r}\right)^{\overline{\alpha}} \qquad \text{for } 0 < r \le R. \qquad \textbf{(WS)}.$$

Define $\nu(r) := (\phi(r) r^d)^{-1}$ for $r > 0$.

- **(WS)** implies $\int_{\mathbb{R}^d}(1 \wedge |x|^2)\nu(|x|)dx < \infty$, so there exists a pure jump isotropic unimodal Lévy process Z with the Lévy measure $\nu(|x|)dx$.

Let $\kappa : (\mathbb{R}^d \times \mathbb{R}^d) \to (0,\infty)$ be symmetric and measurable function such that $L_0^{-1} \le \kappa(x,y) \le L_0$ with some constant $L_0 > 1$.

Let J be a symmetric measurable function. The first set of the conditions on J is following:

(**J1**) **(J1.1)** $J(x,y) := \kappa(x,y)\nu(|x-y|)$ on $|x-y| \le 1$,

(J1.2) $\sup_x \int_{|x-y|>1} J(x,y)dy < \infty$,

(J1.3) For any $M > 0$, there exists $C_M > 1$ such that
$$C_M^{-1}\nu(|x-y|) \le J(x,y) \le C_M \nu(|x-y|) \text{ for } |x-y| < M.$$

Let χ be a nondecreasing function on $(0,\infty)$ with $\chi(r) = \chi(0)$ for $r \in (0,1]$, and let there exist γ_1, γ_2, $L_1, L_2 > 0$ and $\beta \in [0,\infty]$ such that $L_1 e^{\gamma_1 r^\beta} \le \chi(r) \le L_2 e^{\gamma_2 r^\beta}$ for $r > 1$.

The second set of the conditions on J is following:

$$(\mathbf{J2})\ J(x,y) := \kappa(x,y)\nu(|x-y|)\chi(|x-y|)^{-1} = \begin{cases} \kappa(x,y)\left(\phi(|x-y|)|x-y|^d \cdot \chi(|x-y|)\right)^{-1} & \text{if } \beta \in [0,\infty), \\ \kappa(x,y)\left(\phi(|x-y|)|x-y|^d\right)^{-1} \mathbf{1}_{\{|x-y|\le 1\}} & \text{if } \beta = \infty. \end{cases}$$

- Clearly **(J2)** implies **(J1.1)** and **(J1.2)**.
- Moreover, if **(J2)** holds and $\beta \ne \infty$, then **(J1)** holds.

Define the Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated with the jumping kernel J:

$$\mathcal{E}(u,v) = \frac{1}{2}\int\int (u(x)-u(y))(v(x)-v(y)J(x,y)dxdy,$$

and $\mathcal{F} = \{u \in L^2(\mathbb{R}^d) : \mathcal{E}(u,u) < \infty\}$. Under the conditions **(J1.1)** and **(J1.2)**, by Schilling & Uemura [5, Theorem 2.1] and [6, Theorem 2.4], $(\mathcal{E}, \mathcal{F})$ is a regular (symmetric) Dirichlet form on $L^2(\mathbb{R}^d, dx)$. Moreover, the corresponding Hunt process Y is conservative and Y has Hölder continuous transition density $p(t,x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ (See, Fukushima, Oshima & Takeda [3]).

2 Heat Kernel Estimates of Y

Let $a \wedge b := \min\{a, b\}$.

Theorem 1: J satisfies (J1.2) and (J1.3).
For each $M, T > 0$, there exists $c \ge 1$ such that for every $(t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ with $|x-y| < M$,

$$c^{-1}\left([\phi^{-1}(t)]^{-d} \wedge t\nu(x,y)\right) \le p(t,x,y) \le c\left([\phi^{-1}(t)]^{-d} \wedge t\nu(x,y)\right)$$

where $\phi^{-1}(t)$ is the inverse function of $\phi(t)$.

For each $a, \gamma, T > 0$, define a function $F_{a,\gamma,T}(t,r)$ on $(0,T] \times [0,\infty)$ as

$$F_{a,\gamma,T}(t,r) := \begin{cases} [\phi^{-1}(t)]^{-d} \wedge t\nu(r)e^{-\gamma r^\beta} & \text{if } \beta \in [0,1], \\ [\phi^{-1}(t)]^{-d} \wedge t\nu(r) & \text{if } \beta \in (1,\infty] \text{ with } r < 1, \\ t\exp\left\{-a\left(r\left(\log\frac{Tr}{t}\right)^{\frac{\beta-1}{\beta}} \wedge r^\beta\right)\right\} & \text{if } \beta \in (1,\infty) \text{ with } r \ge 1, \\ (t/(Tr))^{ar} = \exp\left\{-ar\left(\log\frac{Tr}{t}\right)\right\} & \text{if } \beta = \infty \text{ with } r \ge 1. \end{cases}$$

Theorem 2: J satisfies (J2).
For each $T > 0$, there exist C_1, c_1 and $c_2 \ge 1$ such that for every $(t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$c_2^{-1}F_{c_1,\gamma_2,T}(t,|x-y|) \le p(t,x,y) \le c_2 F_{C_1,\gamma_1,T}(t,|x-y|).$$

- The upper bound of Theorem 1 comes from Chen, Kim & Song [2, (2.6)].
- Also the upper bound of Theorem 2 comes from Kaleta & Sztonyk [4, Theorem 2, Proposition 1] for $\beta \in [0,1]$ case, and Chen, Kim & Kumagai [1, Theorems 1.2 and 1.4] for $\beta \in (1,\infty]$ case.

3 Dirichlet Heat Kernel Estimates for Y^D

3.1 Assumptions on Open Sets

An open set D in $\mathbb{R}^d$ (when $d \ge 2$) is said to be a $C^{1,\rho}$ open set for $\rho \in (0,1]$: if there exists a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for any $z \in \partial D$, there exists a $C^{1,\rho}$-function $\psi = \psi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying

$$\psi(0) = 0,\ \nabla\psi(0) = (0,\cdots,0),\ \|\nabla\psi\|_\infty \le \Lambda_0, |\nabla\psi(x) - \nabla\psi(y)| \le \Lambda_0|x-y|^\rho$$

and there exists an orthonormal coordinate system CS_z of $z = (z_1,\cdots,z_{d-1},z_d) := (\widetilde{z},\ z_d)$ such that

$$B(z,R_0) \cap D = \{y = (\widetilde{y}, y_d) \in B(0,R_0) \text{ in } CS_z : y_d > \psi(\widetilde{y})\}.$$

The pair (R_0, Λ_0) is called the characteristic of the $C^{1,\rho}$ open set D.

3.2 Condition on the Processes

- Condition on the regularity of $\kappa(x,y)$:

(K$_\eta$) There exist $L_3 > 0$ and $\eta > \overline{\alpha}/2$ such that $|\kappa(x,y) - \kappa(x,x)| \le L_3|x-y|^\eta$ for every $x,\ y \in \mathbb{R}^d$, $|x-y| \le 1$.

- Condition on the regularity of ϕ:

(SD) $\phi \in C^1(0,\infty)$ and $r \to -\nu'(r)/r$ is decreasing.

Let Y^D be a subprocess of Y killed upon leaving D and $p_D(t,x,y)$ be a transition density of Y^D. From now on, we assume that D is a $C^{1,\rho}$ open set ($\rho \in (\overline{\alpha}/2, 1]$) with characteristics (R_0, Λ_0), and the jumping intensity kernel J satisfies **(K$_\eta$)** and **(SD)**.

3.3 Main results

Let $\delta_D(x)$ be a distance between x and D^c, and let $\Psi(t,x) := \left(1 \wedge \frac{\sqrt{\phi(\delta_D(x))}}{\sqrt{t}}\right)$.

Theorem 3: J satisfies (J1).
Suppose D is bounded and $T > 0$.
(1) There exists $c_1 > 0$ such that for any $(t,x,y) \in (0,T] \times D \times D$,

$$c_1^{-1}\Psi(t,x)\Psi(t,y)\,p(t,x,y) \le p_D(t,x,y) \le c_1\Psi(t,x)\Psi(t,y)\,p(t,x,y).$$

(2) There exists $c_2 \ge 1$ such that for any $(t,x,y) \in [T,\infty) \times D \times D$,

$$c_2^{-1}e^{-t\lambda^D}\sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))} \le p_D(t,x,y) \le c_2 e^{-t\lambda^D}\sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))},$$

where $-\lambda^D < 0$ is the largest eigenvalue of the generator of Y^D.

(E) (additional assumption on D) The path distance in an open set U is comparable to the Euclidean distance with characteristic λ_1: if for any x, y in the U, there exists a curve l in U connecting x and y such that $|l| \le \lambda_1|x-y|$.

Theorem 4: J satisfies (J2).
[Small time estimate]
(1) [the upper bound] There exists $c_1 > 0$ such that for any $(t,x,y) \in (0,T] \times D \times D$,

$$p_D(t,x,y) \le c_1\,\Psi(t,x)\Psi(t,y)\begin{cases} F_{C_1\wedge\gamma_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta \in [0,\infty), \\ F_{C_1,\gamma_1,T}(t,|x-y|/6) & \text{if } \beta = \infty, \end{cases}$$

where C_1 is the constant in Theorem 2.
(2) [the lower bound] There exists $c_2 > 0$ such that for any $(t,x,y) \in (0,T] \times D \times D$,

$$p_D(t,x,y) \ge c_2\,\Psi(t,x)\Psi(t,y)\begin{cases} [\phi^{-1}(t)]^{-d} \wedge \frac{t\nu(|x-y|)}{e^{\gamma_2|x-y|^\beta}} & \text{if } \beta \in [0,1], \\ [\phi^{-1}(t)]^{-d} \wedge t\nu(|x-y|) & \text{if } \beta \in (1,\infty)\ \&\ |x-y| < 1, \\ & \text{or } \beta = \infty\ \&\ |x-y| \le 4/5 \end{cases}$$

where $\phi^{-1}(t)$ is the inverse function of $\phi(t)$.
(3) [the lower bound] Suppose that D has the assumption **(E)**. There exist $c_3, c_4 > 0$ such that for every x, y in the same component of D and $t \le T$,

$$p_D(t,x,y) \ge c_3\,\Psi(t,x)\Psi(t,y)\begin{cases} F_{c_4,\gamma_2,T}(t,|x-y|) & \text{if } \beta \in (1,\infty)\ \&\ |x-y| \ge 1, \\ F_{c_4,\gamma_2,T}(t,5|x-y|/4) & \text{if } \beta = \infty\ \&\ |x-y| \ge 4/5. \end{cases}$$

(4) [the lower bound] If $\beta \in (1,\infty)$, there exists $c_5 > 0$ such that for every x, y in the different components of D with $|x-y| \ge 1$,

$$p_D(t,x,y) \ge c_5\,\Psi(t,x)\Psi(t,y)\frac{t\nu(|x-y|)}{e^{\gamma_2(5|x-y|/4)^\beta}}.$$

[Large time estimates for $\beta = \infty$] Suppose that D is bounded and connected. Then Theorem 3(2) hold, i.e., there exists $c_6 \ge 1$ such that for any $(t,x,y) \in [T,\infty) \times D \times D$,

$$c_6^{-1}e^{-t\lambda^D}e^{-t\lambda^D}\sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))} \le p_D(t,x,y) \le c_6 e^{-t\lambda^D}\sqrt{\phi(\delta_D(x))}\sqrt{\phi(\delta_D(y))},$$

where $-\lambda^D < 0$ is the largest eigenvalue of the generator of Y^D.

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