Reflected Brownian motion on nested fractals

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Motivation

K. Pietruska-Pałuba in the paper *The Lifschitz singularity for the density of states on the Sierpinski gasket* (1991) proved the existence of the density of states for the Laplacian on the infinite Sierpinski gasket.
To obtain an upper-bound for the Laplace transform of the density of states, one must reduce the problem to the one on compact state-space (the single fractal complex).
We do this by reflecting the Brownian motion on vertices of fractal complexes.
The estimates for unbounded fractal are obtained when we pass to infinity with the diameter of the complex.

Notation

For $d > 1$ let $\Psi_i : \mathbb{R}^d \to \mathbb{R}^d, 1 \le i \le M$ be similitudes given by formula

$$\Psi_i(x) = (1/L)U(x) + \nu_i,$$

where U is an isometry of $\mathbb{R}^d$, $L > 1$ is a scaling factor, $\nu_i \in \mathbb{R}^d$ (for future calculations we shall assume, that $\nu_1 = 0$).
There exists the unique nonempty compact set $\mathcal{K}^{\langle 0 \rangle}$ such that

$$\mathcal{K}^{\langle 0 \rangle} = \bigcup_{i=1}^{M} \Psi_i\left(\mathcal{K}^{\langle 0 \rangle}\right).$$

The set $\mathcal{K}^{\langle 0 \rangle}$ is the (bounded) fractal.

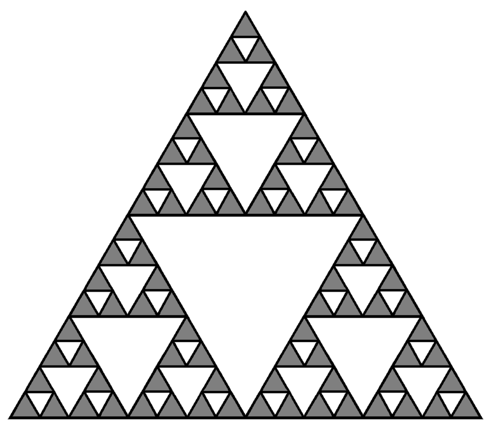

Figure: Sierpiński Triangle: 3 similitudes, $L = 2$

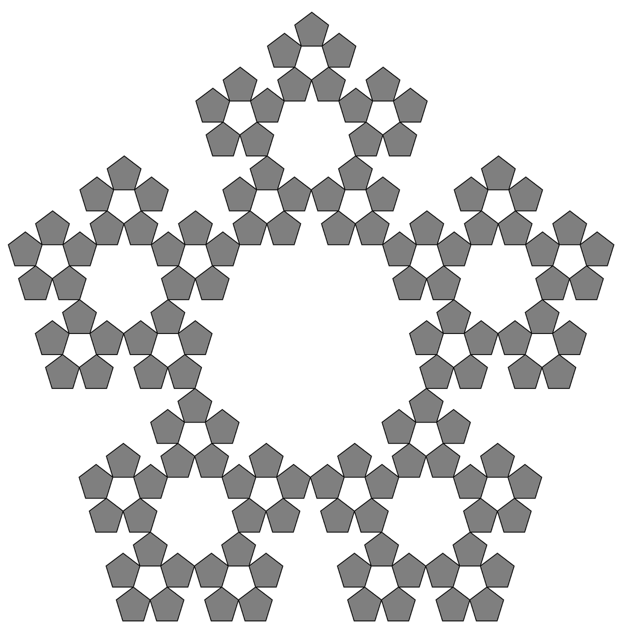

Figure: Sierpiński Pentagon: 5 similitudes, $L = \frac{3+\sqrt{5}}{2}$

Essential fixed points

The fixed point $x \in \mathcal{K}^{\langle 0 \rangle}$ is an essential fixed point if there exists another fixed point $y \in \mathcal{K}^{\langle 0 \rangle}$ and similitudes Ψ_i, Ψ_j such that $\Psi_i(x) = \Psi_j(y)$.
$V_0^{\langle 0 \rangle}$ is the set of the essential fixed points.

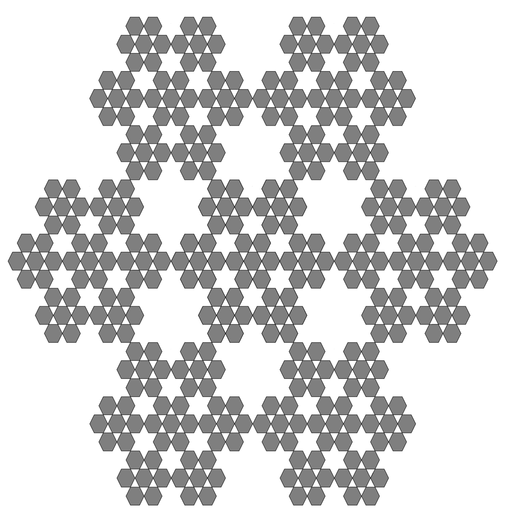

Figure: Lindstrøm snowflake: 7 similitudes, $L = 3$ (7 fixed points, but only 6 essential fixed points).

Nested fractals

The nested fractal is a fractal satisfying the following conditions:

1. $\# V_0^{\langle 0 \rangle} \ge 2$
2. There exists an open set $U \in \mathbb{R}^d$ such that for $i \neq j$ $\Psi_i(U) \cap \Psi_j(U) = \emptyset$ and $\bigcup_i \Psi_i(U) \subseteq U$
3. (Nesting) Let T, S be different 1-complexes. Then $T \cap S = V(T) \cap V(S)$.
4. (Symmetry) For $x, y \in V_0^{\langle 0 \rangle}$ let $R_{x,y}$ denote the symmetry with respect to hyperplane bisecting the segment $[x, y]$. Then
$$\forall i \in \{1, ..., M\}\ \forall x, y \in V_0^{\langle 0 \rangle}\ \exists j \in \{1, ..., M\}\quad R_{x,y}\left(\Psi_i\left(V_0^{\langle 0 \rangle}\right)\right) = \Psi_j\left(V_0^{\langle 0 \rangle}\right)$$
5. (Connectivity) On the set $V_1^{\langle 0 \rangle} = \bigcup_i \Psi_i(V_0^{\langle 0 \rangle})$ we define graph structure E_1 as follows:
$(x, y) \in E_1$ if x and y are in the same 1-complex.
Then the graph $(V_1^{\langle 0 \rangle}, E_1)$ is connected.

Unbounded fractal

We define the unbounded fractal as

$$\mathcal{K}^{\langle \infty \rangle} = \bigcup_{n=0}^{\infty} L^n \mathcal{K}^{\langle 0 \rangle}$$

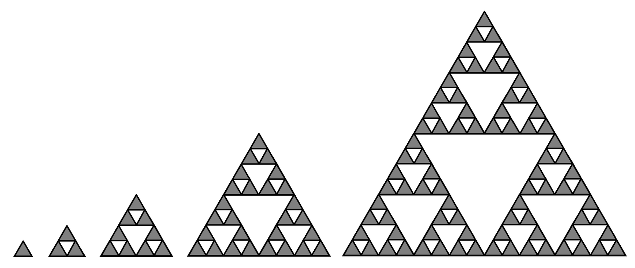

Figure: Four iterations of expanding Sierpiński gasket.

Labelling and projections

We label vertices from $V_0^{\langle 0 \rangle}$ as $a_1, a_2, ..., a_k$.
We expand the labelling on all other vertices from $V_0^{\langle \infty \rangle}$ in such way, that the orientation of labels on each 0-complex is preserved.

For an arbitrary $x \in \mathcal{K}^{\langle \infty \rangle} \backslash V_0^{\langle \infty \rangle}$, x belongs to exactly one 0-complex $\Delta_0(x)$ and x can be written as

$$x = \sum_{i=1}^{k} x_i \cdot a_i(x),$$

where $a_i(x)$ are vertices of $\Delta_0(x)$ (with proper labels) and $x_i \in \mathbb{R}$, $\sum_i x_i = 1$.

We define a projection map from the unbounded fractal onto the primary 0-complex $\mathcal{K}^{\langle 0 \rangle}$ by setting

$$\pi_0(x) = \sum_{i=1}^{k} x_i \cdot a_i(0),$$

where $a_i(0)$ are vertices from $V_0^{\langle 0 \rangle}$ (with proper labels).

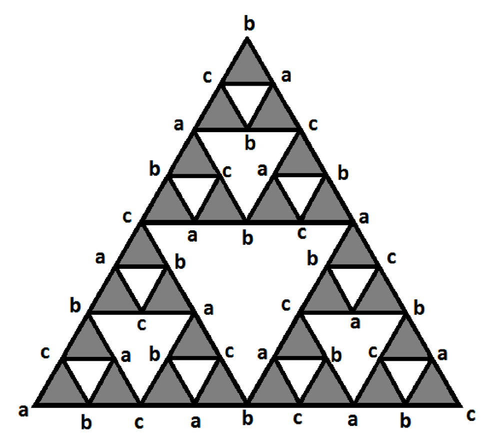

Figure: Labelling of vertices of the Sierpiński gasket.

Reflected process

The reflected Brownian motion on $\mathcal{K}^{\langle 0 \rangle}$ is defined as follows:

$$X_t^0 = \pi_0(Z_t),$$

where Z_t is a diffusion on unbounded fractal $\mathcal{K}^{\langle \infty \rangle}$.

The transition density for this process will be given by:

$$q_0(t, x, y) = \begin{cases} \sum_{y' \in \pi_0^{-1}(y)} p(t, x, y') & \text{if } x, y \in \mathcal{K}^{\langle 0 \rangle}, y \notin V_0^{\langle 0 \rangle} \\ \sum_{y' \in \pi_0^{-1}(y)} p(t, x, y')\text{rank}(y') & \text{if } y \in V_0^{\langle 0 \rangle} \end{cases} \quad (1)$$

where rank(y') is the number of 0-complexes intersecting at point y'.

Theorem (K. Kaleta, MO, K. Pietruska-Pałuba) $\left(X_t^0\right)_{t \ge 0}$ is a continuous strong Markov process.
Its transition density $q_0(t, x, y)$ is continuous and symmetric with respect to x and y.

Impossibility of labelling the Lindstrøm snowflake

Can we consistently label vertices of any nested fractal?
No, there exist fractals which cannot be labelled, e.g. the Lindstrøm snowflake.
Having labelled vertices of the bottom left complex clockwise as a, b, c, d, e, f we know that the bottom right complex must have its left vertex labelled as c. We can label other vertices of this complex clockwise or counter-clockwise. Either way, the middle complex has its two adjacent vertices labelled b and b or b and d, so the order or even uniqueness of labels of vertices in this complex is not preserved.

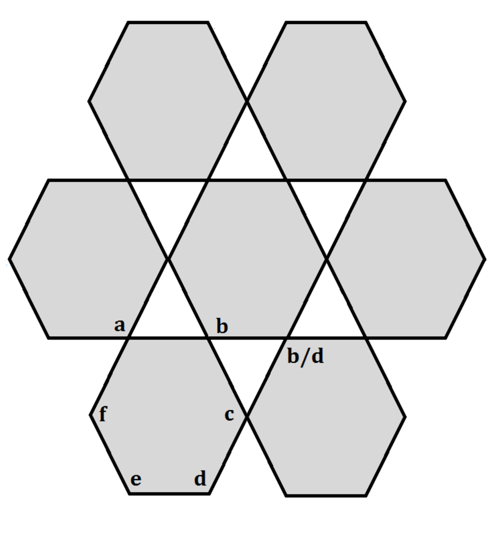

Figure: Illegal labelling of vertices of the Lindstrøm snowflake.

Shape of the complexes

Prepositon. If $d = 2$, then the points from $V_0^{\langle 0 \rangle}$ are vertices of a regular polygon.

Prepositon. If $d = 3$, then the points from $V_0^{\langle 0 \rangle}$ are vertices of a platonic solid.

Sufficient conditions on $\mathbb{R}^2$

Condition 1. If there are 3 essential fixed points (complexes are triangular), the fractal can be labelled.

Condition 2. If there are 4 essential fixed points (complexes are square), the fractal can be labelled.

Condition 3. If there are p essential fixed points, p prime, the fractal can be labelled.

Condition 4. If the number of fixed points is equal to the number of essential fixed points (complexes form a ring structure), the fractal can be labelled.

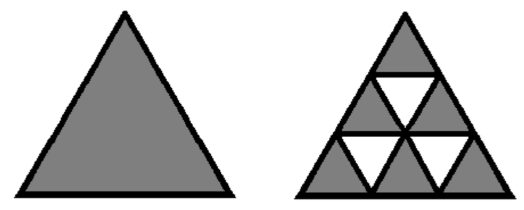

Figure: An example of a fractal with three essential fixed points (description of one iteration).

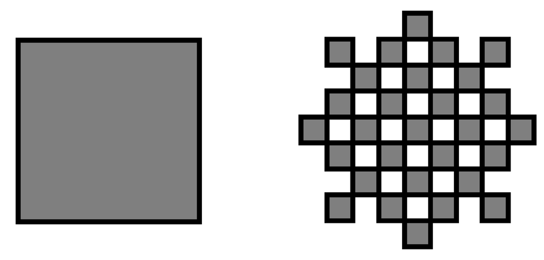

Figure: An example of a fractal with four essential fixed points (description of one iteration).

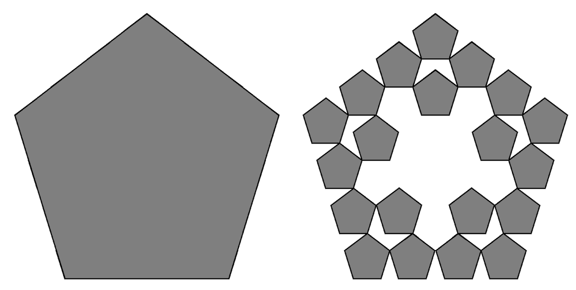

Figure: An example of a fractal with five essential fixed points (description of one iteration).

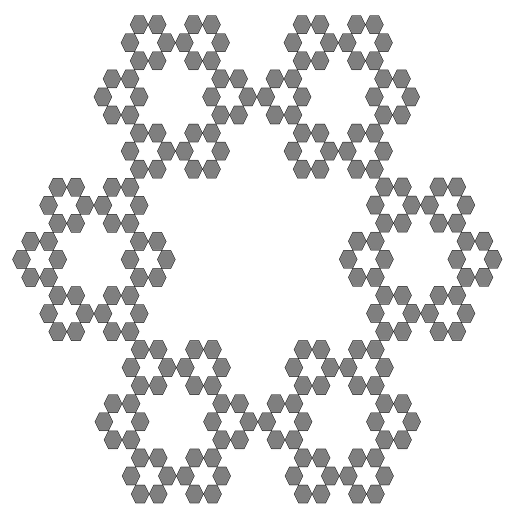

Figure: An example of a fractal with six essential fixed points and six fixed points - complexes creating ring structures (after three iterations).

Sufficient conditions on $\mathbb{R}^3$

Conditions. If there are 4, 6 or 8 essential fixed points (the complexes are tetrahedral, cubic or octahedral), the fractal can be labelled.

We suppose that other fractals (with dodecahedral or icosahedral complexes) also can be labelled, but we have not yet constructed any fractal of that shape.

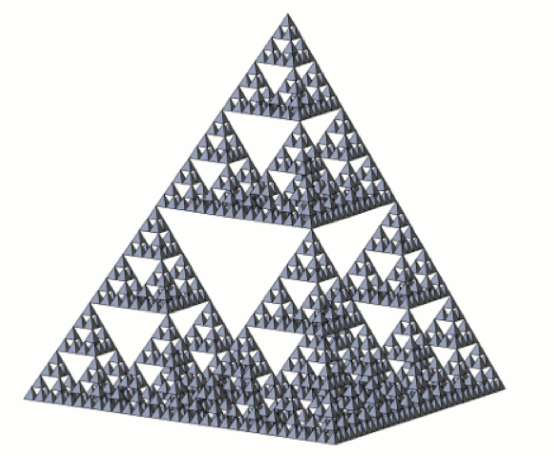

Figure: An example of a 3D fractal with four essential fixed points: The Sierpiński pyramid (after five iterations).

Bibliography

[1]: K. Pietruska-Pałuba *The Lifschitz singularity for the density of states on the Sierpinski gasket*, Probab. Theory Related Fields 89 (1991), no. 1, 1-33.

[2]: K. Kaleta, M. Olszewski, K. Pietruska-Pałuba *Reflected processes on nested fractals*, preprint (2016).