Multi-type continuous-state branching processes

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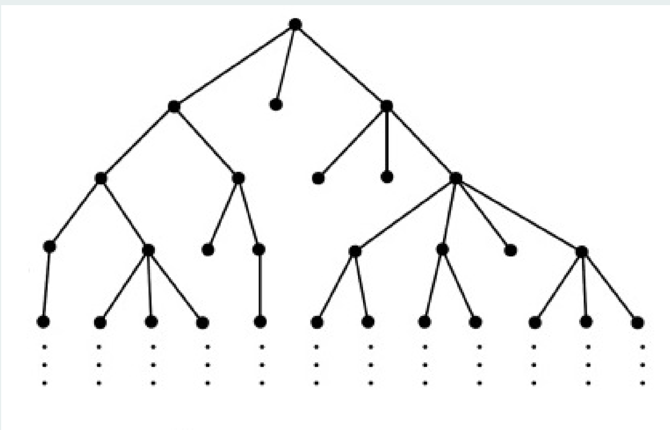
Introduction

A *continuous-state branching process* is a $[0,\infty]$-valued strong Markov process $X=\{X_t : t\geq 0\}$ with probabilities $\{\mathbb{P}_x : x\geq 0\}$ such that for any $x,y\geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$. i.e. X satisfies the branching property.



Galton-Watson process: Discrete analogous

The dynamics of X are characterized by its *branching mechanism*; a function $\psi:[0,\infty)\to\mathbb{R}$ that satisfies the Lévy-Khintchine formula

$$\psi(\lambda)=-a\lambda+\gamma^2\lambda^2+\int_{(0,\infty)}(e^{-\lambda x}-1+\lambda x \mathbf{1}_{\{x<1\}})\mu(dx),$$

where $a\in\mathbb{R}$, $\gamma\geq 0$ and μ is a measure concentrated on $(0,\infty)$ such that

$$\int_{(0,\infty)}(1\wedge x^2)\mu(dx)<\infty.$$

Specifically,

$$\mathbb{E}_x\left[e^{-\lambda X_t}\right]=\exp\{-xu_t(\lambda)\},\qquad \text{for } \lambda\geq 0,$$

where $u_t(\lambda)$ is determined by the integral equation

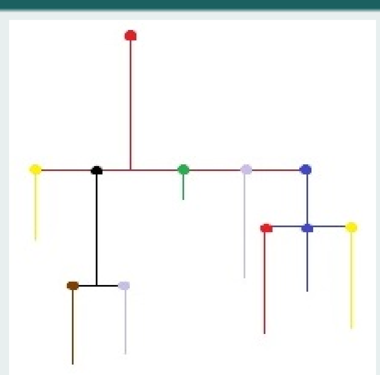
$$u_t(\lambda)=\lambda-\int_0^t\psi(u_s(\Lambda))\mathrm{d}s,\qquad \lambda,t\geq 0.$$

Let $\mathcal{E}:=\{\lim_{t\to\infty}X_t=0\}$ be the event of extinction. Then, $\mathbb{P}_x(\mathcal{E})=1$ for all $x\geq 0$ if and only if $\psi'(0+)\geq 0$. Moreover, for $x>0$

$$\mathbb{P}_x(\mathcal{E})=e^{-\phi(0)x},$$

where, $\phi(0)=\sup\{\lambda\geq 0:\psi(0)=0\}$.

Characteristics in the new process



Multi-type Galton-Watson process: Discrete analogous

- Infinite countable number of types ($\mathbb{N}$).
- Continuous time.
- $[0,\infty)^{\mathbb{N}}$-valued.
- Strong Markov process.
- Branching property with local and non-local branching mechanism.

Let $\mathcal{B}(\mathbb{N})$ be the space of bounded Borel functions on $\mathbb{N}$ and $\mathcal{M}(\mathbb{N})$ be the space of finite Borel measures on $\mathbb{N}$. For $f\in\mathcal{B}(\mathbb{N})$ and $\mathcal{M}(\mathbb{N})$ denote

$$\langle f,\mu\rangle:=\sum_{i\geq 1}f(i)\mu(i).$$

Multi-type continuous-state branching processes

A Multi-type continuous-state branching process is a $[0,\infty)^{\mathbb{N}}$-valued strong Markov process $X=(X_t:t\geq 0)$ with probabilities $\{\mathbf{P}_\mu,\mu\in\mathcal{M}(\mathbb{N})\}$ that satisfies the branching property: for all $\mu,\nu\in\mathcal{M}(\mathbb{N})$,

$$\mathbf{E}_{\mu+\nu}[\mathrm{e}^{-\langle f,X_t\rangle}]=\mathbf{E}_\mu[\mathrm{e}^{-\langle f,X_t\rangle}]\mathbf{E}_\nu[\mathrm{e}^{-\langle f,X_t\rangle}],\qquad f\in\mathcal{B}^+(N),\ t\geq 0.$$

In particular,

$$\mathbf{E}_\mu[\mathrm{e}^{-\langle f,X_t\rangle}]=\exp\{-\langle V_tf,\mu\rangle\},$$

where, for $i\in\mathbb{N}$,

$$V_tf(i)=f(i)-\int_0^t\left[\psi(i,V_sf(i))+\phi(i,V_sf)\right]\mathrm{d}s,\qquad t\geq 0.$$

with branching mechanisms ψ and ϕ given by:

Branching mechanisms

Local mechanism $\psi:\mathbb{N}\times[0,\infty)\to\mathbb{R}$.

$$\psi(i,z)=b(i)z+c(i)z^2+\int_0^\infty(\mathrm{e}^{-zu}-1+zu)\ell(i,\mathrm{d}u),$$

where $b\in\mathcal{B}(\mathbb{N})$, $c\in\mathcal{B}^+(\mathbb{N})$ and, for each $i\in\mathbb{N}$, $(u\wedge u^2)\ell(i,\mathrm{d}u)$ is a bounded kernel from $\mathbb{N}$ to $(0,\infty)$.

Non-local mechanism $\phi:\mathbb{N}\times\mathcal{B}^+(\mathbb{N})\to\mathbb{R}$.

$$\phi(i,f)=-\beta(i)\left[d(i)\langle f,\pi_i\rangle+\int_0^\infty(1-\mathrm{e}^{-u\langle f,\pi_i\rangle})\mathrm{n}(i,\mathrm{d}u)\right],$$

where $d,\beta\in\mathcal{B}^+(\mathbb{N})$ and, for $i\in\mathbb{N}$, π_i is a probability distribution on $\mathbb{N}\backslash\{i\}$ and $un(i,\mathrm{d}u)$ is a bounded kernel from $\mathbb{N}$ to $(0,\infty)$.

Intuition

$X_t(i)$ evolves, in part from a local contribution which is that of a continuous-state branching process with mechanism $\psi(i,z)$, but also from a non-local contribution from other types. The mechanism $\phi(i,\cdot)$ dictates how this occurs. Roughly speaking, each type $i\in\mathbb{N}$ seeds an infinitesimally small mass continuously at rate $\beta(i)d(i)\pi_i(j)$ on to sites $j\neq i$ (recall $\pi_i(i)=0$, $i\in\mathbb{N}$). Moreover, it seeds an amount of mass $u>0$ at rate $\beta(i)\mathrm{n}(i,\mathrm{d}u)$ to sites $j\neq i$ in proportion given by $\pi_i(j)$.

Linear semigroup

Define the matrix $M(t)$ by

$$M(t)_{ij}:=\mathbf{E}_{\delta_i}[X_t(j)],\qquad i,j\in\mathbb{N},t\geq 0,$$

and observe that for all $f\in\mathcal{B}^+(\mathbb{N})$, the linear semigroup satisfies

$$\mathbf{E}_{\delta_i}[\langle f,X_t\rangle]=[M(t)f]_i,\qquad t\geq 0.$$

Suppose that M is irreducible. (for any $i,j\in\mathbb{N}$, there exists $t>0$ such that $M_{ij}(t)>0$). Then, the value

$$\Lambda=\sup\left\{\lambda\geq-\infty:\int_0^\infty\mathrm{e}^{\lambda t}M(t)_{ij}\mathrm{d}t<\infty\right\},$$

does not depend on i and j. It is call the spectral radius of M.

Main result: Local extinction dicotomy.

Let define the events

- Local extinction at a finite set of states $A\subset\mathbb{N}$,
$$\mathcal{L}_A:=\{\lim_{t\to\infty}\langle 1_A,X_t\rangle=0\},$$
- Global extinction
$$\mathcal{E}:=\{\lim_{t\to\infty}\langle 1,X_t\rangle=0\}.$$

Theorem

Fix $\mu\in\mathcal{M}(\mathbb{N})$ such that $\sup\{n:\mu(n)>0\}<\infty$. Moreover, suppose that

$$\sup_{i\in\mathbb{N}}\int_1^\infty(x\log x)\ell(i,\mathrm{d}x)+\sup_{i\in\mathbb{N}}\int_1^\infty(x\log x)\mathrm{n}(i,\mathrm{d}x)<\infty,$$

holds.

(i) For any finite set of states $A\subseteq\mathbb{N}$, $\mathbf{P}_\mu(\mathcal{L}_A)=1$ if and only if $\Lambda\geq 0$.

(ii) The vector $v_A(i)=-\log\mathbf{P}_{\delta_i}(\mathcal{L}_A)$, $i\in\mathbb{N}$ is a solution for

$$\psi(i,v_A(i))+\phi(i,v_A)=0,\qquad i\in\mathbb{N}.\tag{1}$$

In addition, the vector $w(i)=-\log\mathbf{P}_{\delta_i}(\mathcal{E})$, $i\in\mathbb{N}$ is also a solution of (1).

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