The sensitivity of the solution to a Lévy driven SDE with respect to perturbations of the noise

Tetiana Kosenkova, Potsdam University

The model

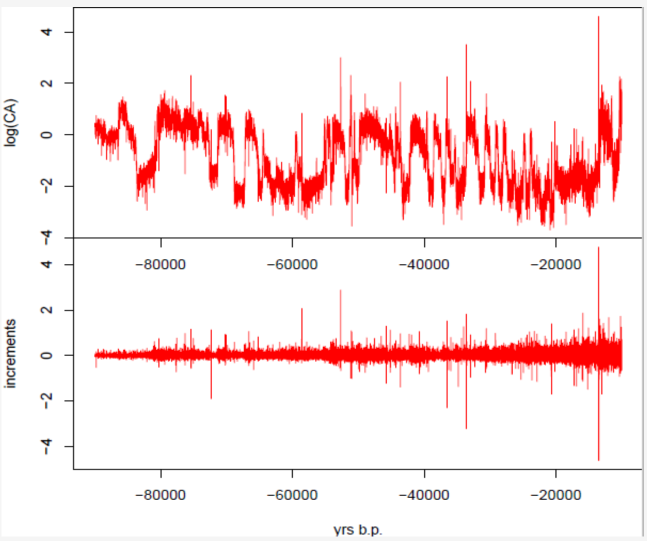
Consider the SDE with Lévy noise, i.e. the process of the following type

$$X(t) = x + \int_0^t V(X(s))\,ds + Z(t) \quad t \geq 0,\ x \in \mathbb{R}, \qquad (1)$$

where the function $V:\mathbb{R}\to\mathbb{R}$ is smooth enough and $(Z(t))_{t\geq 0}$ is a Lévy process. It is uniquely determined by a characteristic triplet (a,b,Π) and satisfies the *Itô-Lévy representation*: $Z_j(t) = at + \sqrt{b}W(t) + \int_0^t \int_{|u|\leq 1} u\tilde{\nu}(ds,du) + \int_0^t \int_{|u|>1} u\,\nu(ds,du), t \geq 0,$ where ν is a Poisson point measure with an intensity measure $\lambda\Pi$ and λ is the Lebesgue measure.

Background

In climatology the SDE's of the type (1) are used to describe *climate dynamics of the oceanic flow* [Ditlevsen, Geophys. Res. Lett. 1999].



In such models function V is considered as the gradient of an "energy potential" $U:\mathbb{R}\to\mathbb{R}$ with two local minima such that $V = -U'$ which correspond to two climate equilibrium states.

Model selection problem

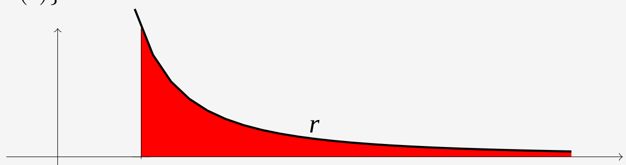
Question: how to verify the hypothesis that the available data follow a model described by the SDE (1)?

Solution: estimate a distance between two solutions to the SDE (1) with different noises in terms of a distance between the characteristic triplets and use it for "goodness-of-fit" test for the model, [Gairing, Högele, K, Kulik, Springer INdAM Volume, 2016]

Question: how to measure a distance between the characteristic triplets, i.e. between Lévy measures?

"Heads" and "Tails" for Lévy measures

For a given Lévy measure Π and given $r>0$ we want a decomposition of Π into two σ-finite measures $\Pi^{H,r}$ and $\Pi^{T,r}$ of the form $\Pi = \Pi^{H,r} + \Pi^{T,r}$, such that the total mass of $\Pi^{T,r}$ is r and there exists $\varepsilon(r)\geq 0$ for which $\mathrm{supp}(\Pi^{H,r}) = \{u : |u| \leq \varepsilon(r)\}$ and $\mathrm{supp}(\Pi^{T,r}) = \{u : |u| \geq \varepsilon(r)\}$



Coupling distance

For Lévy measures Π_1,Π_2 and $r>0$ define the probability measure $\pi^r = \frac{1}{r}\Pi^{T,r}$ and the quantities

$$T(\Pi_1,\Pi_2) := \sup_{r>0} r^{1/2} W_{2,\rho}(\pi_1^r,\pi_2^r), r>0.$$

We shall call T a *coupling distance* on the set of Lévy measures. Here $W_{2,\rho}$ is a coupling-type *Wasserstein-Kantorovich-Rubinstein metric* of order 2, with the metric ρ on $\mathbb{R}$ defined by $\rho(x,y) = |x-y|\wedge 1,\ x,y\in\mathbb{R}$. Below, for $W_{2,\zeta}$ we will use a metric ζ on the space of càdlàg paths $\mathbb{D}(0,1)$, defined by $\zeta(x,y) = \sup_{t\in[0,1]}\rho(x(t),y(t))$.

Proposition The function T is a metric on the set of Lévy measures.

Deviation between the solutions to SDE's

Consider two Lévy processes $(Z_j(t))_{t\geq 0}$, $j=1,2$ and the solutions to the SDE's $X_j(t), j=1,2$ of type (1) and the function $V:\mathbb{R}\to\mathbb{R}$ is *one-sided Lipschitz*, i.e. $(V(x)-V(y))(x-y) \leq L(x-y)^2, \quad x,y\in\mathbb{R}$ for some $L>0$.

Theorem Let (a_j,b_j,Π_j) be two characteristic triplets and $x_j\in\mathbb{R}$ given initial values, $j=1,2$. Then, for any two solutions X_j to SDE's of the type (1) the following holds

$$W^2_{2,\zeta}\Big(\mathrm{Law}(X_1),\mathrm{Law}(X_2)\Big) \leq 2(\rho^2(x_1,x_2)+Q)e^{2.16\,L} + Q_1,$$

where $Q = |a_1-a_2| + (\sqrt{b_1}-\sqrt{b_2})^2 + 4T^2(\Pi_1,\Pi_2) + 2(\Pi_1(|u|>1)+\Pi_2(|u|>1))^{1/2}T(\Pi_1,\Pi_2),$

$Q_1 = 3\sqrt{(\sqrt{b_1}-\sqrt{b_2})^2 + T^2(\Pi_1,\Pi_2)}.$

Reference

Coupling distances between Lévy measures and applications to noise sensitivity of SDE, J. Gairing, M. Högele, T. Kosenkova, A. Kulik, Stoch. Dyn., 15(2) 2015.