Space-time fractional Dirichlet problems

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1 Introduction

In this research project, we establish explicit solutions to a broad class of time-fractional Cauchy problems

$$\partial_t^\beta u(x,t) = Lu(x,t); \quad u(0) = f(x)$$

on a regular bounded domain Ω in d-dimensional Euclidean space, where ∂_t^β is the Caputo fractional derivative of order $0 < \beta < 1$ and L is the semigroup generator of some Markov process on $\mathbf{R}^d$. In particular, we allow the operator L to be nonlocal in space. One important outcome of this research is to describe the appropriate version of these nonlocal operators on a bounded domain. Our method of proof uses a fundamental result of [1] from the theory of semigroups, along with some ideas from the theory of Markov processes. This probabilistic method also establishes stochastic solutions for these equations, i.e., we describe a stochastic process whose probability density functions solve the time-fractional and space-nonlocal diffusion problem on the bounded domain.

2 Killed Feller processes

Let X_t be a Feller process in $\mathbf{R}^d$. That is, for any $x \in \mathbf{R}^d$, the linear operators defined by $P_t f(x) := \mathbf{E}^x[f(X_t)]$ form a strongly continuous, contraction semigroup on $C_0(\mathbf{R}^d)$. The infinitesimal generator of X_t is defined by

$$Lf = \lim_{t \searrow 0} \frac{P_t f - f}{t} \quad \text{in } C_0(\mathbf{R}^d), \tag{1}$$

with the domain $\mathcal{D}(L) \subset C_0(\mathbf{R}^d)$. If $C_c^\infty(\mathbf{R}^d) \subset \mathcal{D}(L)$, then [3] shows that for any $f \in C_0^2(\mathbf{R}^d)$ we have

$$\begin{aligned} Lf(x) = PDO[f](x) := &-c(x)f(x) + l(x)\cdot\nabla f(x) + \nabla\cdot Q(x)\nabla f(x) \\ &+ \int_{\mathbf{R}^d\setminus\{0\}} \left(f(x+y) - f(x) - \nabla f(x)\cdot y I_{B_1}(y)\right) N(x,dy) \end{aligned} \tag{2}$$

for some $c(x) \ge 0$, $l(x) \in \mathbf{R}^d$, $Q(x) \in \mathbf{R}^{d\times d}$ symmetric and positive definite, $N(x,\cdot)$ a positive measure satisfying

$$\int_{\mathbf{R}^d\setminus\{0\}} \min(|y|^2, 1) N(x,dy) < \infty,$$

and B_1 the unit ball. Let $\Omega \subset \mathbf{R}^d$ be a bounded domain. We define the killed process on Ω by

$$X_t^\Omega = \begin{cases} X_t, & t < \tau_\Omega, \\ \partial, & t \ge \tau_\Omega, \end{cases}$$

where $\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}$ and ∂ denotes a cemetery point. We say that Ω is regular if every boundary point of Ω satisfies

$$\mathbf{P}^x(\tau_\Omega = 0) = 1.$$

A Markov process X_t on $\mathbf{R}^d$ or its semigroup P_t is *strong Feller* if for any bounded measurable real-valued function f with compact support on $\mathbf{R}^d$, $P_t f(x)$ is bounded and continuous on $\mathbf{R}^d$. We say that a Feller process (resp., semigroup) is *doubly Feller* if it also has the strong Feller property.

3 The generator of a killed Feller process

Let X_t be a doubly Feller process in $\mathbf{R}^d$ and let $\Omega \subset \mathbf{R}^d$ be a regular bounded domain. We denote by $C_0(\Omega)$ the set of continuous real-valued functions on Ω that tend to zero as $x \in \Omega$ approaches the boundary. It follows from [2] that $P_t^\Omega f(x) := \mathbf{E}^x[f(X_t^\Omega)]$ defines a Feller semigroup on $C_0(\Omega)$. Let L_Ω be the infinitesimal generator of P_t^Ω.

Theorem 1. *The domain of L_Ω is given by*

$$\mathcal{D}(L_\Omega) = \{f \in C_0(\Omega) : L^\sharp f \in C_0(\Omega)\}, \tag{3}$$

where $L^\sharp f$ denotes the pointwise limit in (1) *applied to the zero extension of $C_0(\Omega)$. Also $L_\Omega f(x) = L^\sharp f(x)$ for all $x \in \Omega$, and the convergence to $L^\sharp f$ holds uniformly on compact subsets of Ω.*

Next we show that functions in $\mathcal{D}(L_\Omega)$ can be characterized as functions in $C_0(\Omega)$ that are locally in the domain of L. This will be used for explicitly computing the generator L_Ω.

Theorem 2. *We have*

$$\begin{aligned} \mathcal{D}(L_\Omega) = \{f \in C_0(\Omega) : \exists g \in C_0(\Omega), (f_n) \subset \mathcal{D}(L) \text{ we have } f_n \to f \text{ in } C_0(\mathbf{R}^d) \\ \text{and } Lf_n \to g \text{ uniformly on compact subsets of } \Omega\}, \end{aligned} \tag{4}$$

and for f, g as in (4) *we have $L_\Omega f = g$.*

Finally, we show that we can evaluate the generator $L_\Omega f(x)$ of the killed Feller process pointwise for $x \in \Omega$ using the explicit formula (2) for $Lf(x)$. Let $C_0^2(\Omega)$ denote the set of $C_0(\Omega)$ functions with first and second order partial derivatives that are continuous at every $x \in \Omega$.

Theorem 3. *Suppose that $C_0^2(\mathbf{R}^d)$ is a core of L, so that $Lf(x) = PDO[f](x)$ for every $x \in \mathbf{R}^d$ and $f \in C_0^2(\mathbf{R}^d)$. Then:*

- *For every $f \in \mathcal{D}(L_\Omega)$ there exists $f_n \in C_0^2(\Omega)$ such that $f_n \to f$ uniformly and $PDO[f_n]$ converges uniformly on compact subsets of Ω to $L_\Omega f$.*
- *If $f_n \in C_0^2(\Omega)$ is such that $f_n \to f \in C_0(\Omega)$ uniformly and $PDO[f_n] \to g \in C_0(\Omega)$ converges uniformly on compact subsets of Ω, then $f \in \mathcal{D}(L_\Omega)$ and $L_\Omega f = g$.*

In particular, if $f \in C_0^2(\Omega)$ and $PDO[f] \in C_0(\Omega)$, then $f \in \mathcal{D}(L_\Omega)$ and $L_\Omega f(x) = PDO[f](x)$ for every $x \in \Omega$.

4 Fractional derivatives

The positive and negative Riemann-Liouville fractional derivatives are defined by

$$\mathbb{D}_{[L,x]}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_L^x f(y)(x-y)^{n-\alpha-1}dy,$$
$$\mathbb{D}_{[x,R]}^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^R f(y)(y-x)^{n-\alpha-1}dy$$

for any non-integer $\alpha > 0$ and any $-\infty \le L < x < R \le \infty$, where $n-1 < \alpha < n$. The positive and negative Caputo fractional derivatives are defined by

$$\partial_{[L,x]}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_L^x f^{(n)}(y)(x-y)^{n-\alpha-1}dy,$$
$$\partial_{[x,R]}^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^R f^{(n)}(y)(y-x)^{n-\alpha-1}dy.$$

If $0 < \beta < 1$, then it follows by the uniqueness of the Laplace transform that

$$\partial_{[0,t]}^\beta f(t) = \mathbb{D}_{[0,t]}^\beta f(t) - \frac{t^{-\beta}}{\Gamma(1-\beta)}f(0). \tag{5}$$

Using Theorem 3 we can compute the generator of a killed stable process X_t on an interval in terms of fractional derivatives.

Theorem 4. *Let X_t be any stable Lévy process with index $1 < \alpha < 2$ and let $\Omega = (L,R)$. Then for all $x \in \Omega$ and any $f \in C_0^2(\Omega)$ such that $PDO[f] \in C_0(\Omega)$ we have*

$$L_\Omega f(x) = -af'(x) + b\,\mathbb{D}_{[L,x]}^\alpha f(x) + c\,\mathbb{D}_{[x,R]}^\alpha f(x).$$

5 Applications to fractional Cauchy problems

Let $g_\beta(u)$ denote the probability density function of the standard stable subordinator, with the Laplace transform

$$\int_0^\infty e^{-su} g_\beta(u)du = e^{-s^\beta}$$

for some $0 < \beta < 1$. Suppose that D_t is a Lévy process such that $g_\beta(u)$ is the probability density of D_1, and define the *inverse stable subordinator* (first passage time)

$$E_t = \inf\{u > 0 : D_u > t\}.$$

Then [1] implies that the function

$$v(x,t) := \int_0^\infty g_\beta(r) P_{(t/r)^\beta}^\Omega f(x)\,dr$$

is the unique solution to the time-fractional Cauchy problem

$$\mathbb{D}_t^\beta v(x,t) = L_\Omega v(x,t) + \frac{t^{-\beta}}{\Gamma(1-\beta)}f(0); \quad v(x,0) = f(x)$$

for any $f \in \mathcal{D}(L_\Omega)$. Using (5), it follows that the same function also solves

$$\partial_t^\beta v = L_\Omega v; \quad v(0) = f$$

for any $f \in \mathcal{D}(L_\Omega)$. Since

$$h(w,t) = \frac{t}{\beta} w^{-1-1/\beta} g_\beta(tw^{-1/\beta})$$

is the probability density function of the inverse stable subordinator E_t, it follows by a simple change of variables that

$$v(x,t) = \int_0^\infty h(w,t)P_w^\Omega f(x)\,dw = \int_0^\infty u(x,w)h(w,t)\,dw = \mathbf{E}^x[f(X_{E_t}^\Omega)].$$

Example. Let X_t be any stable Lévy process with index $1 < \alpha < 2$ and let $\Omega = (L,R)$. By Theorem 3 and Theorem 4, L_Ω is the unique closed extension of $-a\partial_x + b\,\mathbb{D}_{[L,x]}^\alpha + c\,\mathbb{D}_{[x,R]}^\alpha$. Then the function $u(x,t) = \mathbf{E}^x[f(X_t)I\{\tau_\Omega < t\}]$ is the unique solution to the space-fractional Dirichlet problem

$$\begin{aligned} &\partial_t u(x,t) = -a\partial_x u(x,t) + b\,\mathbb{D}_{[L,x]}^\alpha u(x,t) + c\,\mathbb{D}_{[x,R]}^\alpha u(x,t) \quad \forall\, x \in \Omega,\ t > 0 \\ &u(x,0) = f(x) \quad \forall\, x \in \Omega; \\ &u(x,t) = 0 \quad \forall\, x \notin \Omega, t \ge 0 \end{aligned} \tag{6}$$

for any $f \in \mathcal{D}(L_\Omega)$, and the unique mild solution to (6) for any $f \in C_0(\Omega)$. Also, for any $0 < \beta < 1$ the function $v(x,t) = \mathbf{E}^x[f(X_{E_t}^\Omega)]$ is the unique solution to the space-time fractional Dirichlet problem

$$\begin{aligned} &\partial_t^\beta v(x,t) = -a\partial_x v(x,t) + b\,\mathbb{D}_{[L,x]}^\alpha v(x,t) + c\,\mathbb{D}_{[x,R]}^\alpha v(x,t) \quad \forall\, x \in \Omega,\ t > 0 \\ &v(x,0) = f(x) \quad \forall\, x \in \Omega; \\ &v(x,t) = 0 \quad \forall\, x \notin \Omega, t \ge 0 \end{aligned} \tag{7}$$

for any $f \in \mathcal{D}(L_\Omega)$, and the unique mild solution to (7) for any $f \in C_0(\Omega)$.

References

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