Invariance principle for random walks with anomalous recurrence properties

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Joint work with
Nicholas Georgiou and Andrew Wade

Outline

1. From classical to nonhomogeneous random walk
2. Elliptical random walk
3. Diffusion limits

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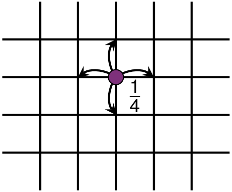
Classical zero-drift random walks

1. Symmetric simple random walk on $\mathbb{Z}^d$

- $X_n \in \mathbb{Z}^d$, $X_0 = 0$.
- Given $X_0, \ldots, X_n$, new location X_{n+1} is uniformly distributed on the 2d adjacent lattice sites to X_n.

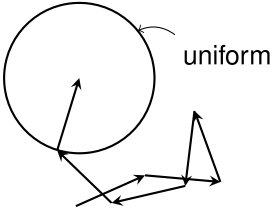
> **Theorem (Pólya 1921)**
>
> *SRW is recurrent if $d = 1$ or $d = 2$, but transient if $d \geq 3$.*



Classical zero-drift random walks

2. Pearson–Rayleigh random walk in $\mathbb{R}^d$

- $X_n \in \mathbb{R}^d$, $X_0 = 0$.
- Given $X_0, \ldots, X_n$, new location X_{n+1} is uniformly distributed on the unit circle/sphere centred at X_n.



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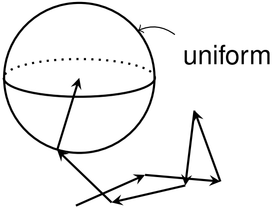

Recurrence/transience of homogeneous random walks

Let (X_n) be a spatially homogeneous random walk in $\mathbb{R}^d$.

So X_{n+1} depends only on X_n, but $\Delta := X_{n+1} - X_n$ is independent of X_n (and n).

Let $\mu = \mathbb{E}\Delta$, the mean drift vector of the random walk.

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Theorem (Chung–Fuchs)

Under mild conditions, if $\mu = 0 \in \mathbb{R}^d$, then (X_n) is

- *recurrent if $d = 1$ or $d = 2$;*
- *transient if $d \geq 3$.*

This result applies both to the symmetric simple RW and the Pearson–Rayleigh RW.

Definition

- recurrence: $\mathbb{P}[\text{return to (nbrhood of) origin}] = 1$.
- transience: $\mathbb{P}[\text{return to (nbrhood of) origin}] < 1$.

Scaling limit for homogeneous random walks

Under mild non-degeneracy conditions (non-singularity of $\mathbb{E}[\Delta\Delta^\top]$), we have (up to a linear transformation):

Theorem (Donsker)

Spatially homogeneous random walk in $\mathbb{R}^d$ with zero drift converges to d-dimensional Brownian motion after diffusive scaling:

$$\left(\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}}\right)_{t\in[0,1]} \Longrightarrow (b_t)_{t\in[0,1]} .$$

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Brownian motion on $\mathbb{R}^d$ ($d \geq 2$) possesses a skew-product representation.

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Non-homogeneous random walks

What if we allow $\Delta = X_{n+1} - X_n$, the jump distribution, to depend on the current location?

Then $\mu(x) := \mathbb{E}_x \Delta := \mathbb{E}[\Delta \mid X_n = x]$ becomes a function of the current position $x \in \mathbb{R}^d$.

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Question

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Answer

For $d = 1$: yes (essentially) — zero drift implies recurrence.
For higher dimensions: no — either behaviour is possible.

Theorem

There exist non-homogeneous random walks with $\mu(x) = 0$ for all $x \in \mathbb{R}^d$ that are

- *transient in $d = 2$;*
- *recurrent in $d \geq 3$.*

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We modify the Pearson–Rayleigh random walk to make jumps distributed on an ellipse.

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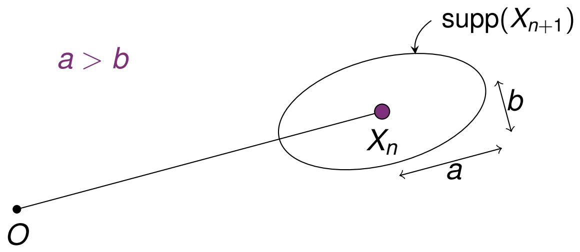

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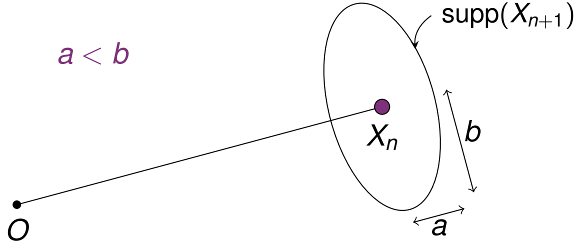

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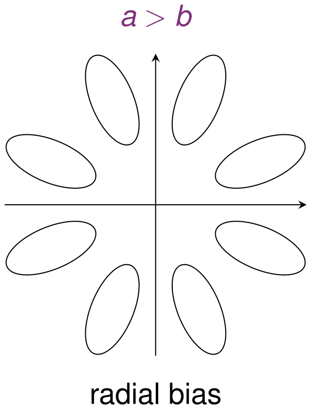
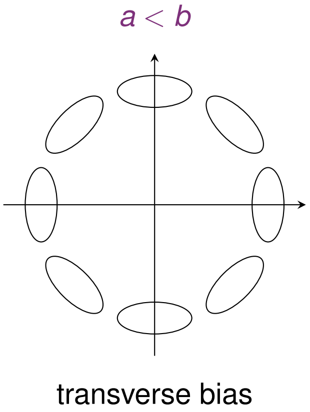
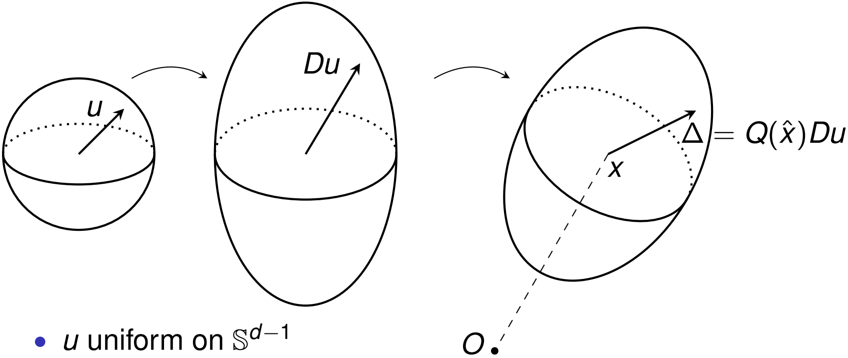




Elliptical random walk ($d \geq 2$)

Suppose $X_n = x \in \mathbb{R}^d$. Write $\hat{x}$ for unit vector in direction x.



- u uniform on $\mathbb{S}^{d-1}$
- $D = \text{diag}(a, b, \ldots, b)$
- $Q(\hat{x})$ orthogonal matrix, with $Q(\hat{x})e_1 = \hat{x}$.

Moments of Δ

Notation: write $\mathbb{E}_x[\,\cdot\,]$ for $\mathbb{E}[\,\cdot \mid X_n = x]$ and write Δ_x for the component of Δ in direction x:

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Therefore, by construction,

$$\mathbb{E}_x[\Delta] = 0, \quad \mathbb{E}_x[\Delta\Delta^\top] = \frac{1}{d} Q(\hat{x}) D^2 Q^\top(\hat{x}).$$

Hence,

$$\mathbb{E}_x[\Delta_x] = 0, \quad \mathbb{E}_x[\Delta_x^2] = \frac{a^2}{d}, \quad \mathbb{E}_x[\|\Delta\|^2] = \frac{a^2 + (d-1)b^2}{d}.$$

Radial component of X_n

We analyse (X_n) by considering $R_n := \|X_n\|$.

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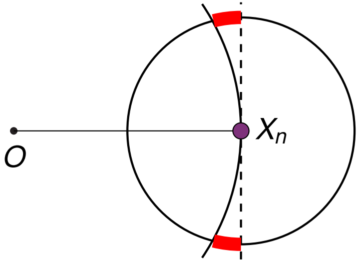
Moreover, it has asymptotically zero drift:

$$\mathbb{E}[R_{n+1} - R_n \mid R_n = r] \sim c/r,$$

where positive constant c depends on model parameters and ambient dimension.



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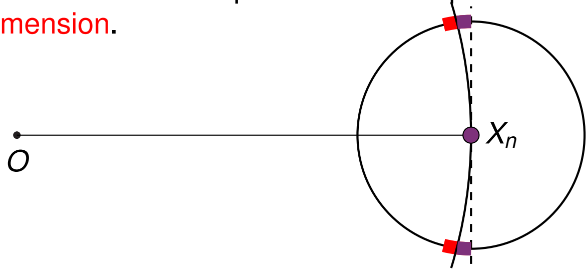

Lamperti's classification

Define $\mu_k(r) := \mathbb{E}[(R_{n+1} - R_n)^k \mid R_n = r]$.

In the early 1960s, John Lamperti studied in detail how the asymptotics of a stochastic process on $\mathbb{R}_+$ are determined by the first two moment functions of its increments, μ_1 and μ_2.

Theorem (Lamperti, 1960)

Let (R_n) be a Markov chain on $\mathbb{R}_+$. Under mild conditions:

- *If $2r\mu_1(r) - \mu_2(r) > 0$ for all large enough r, then R_n is transient,*
- *If $2r\mu_1(r) - \mu_2(r) < 0$ for all large enough r, then R_n is recurrent.*

Recurrence/transience of elliptical random walk

Given $X_n = x$,

$$\begin{aligned} R_{n+1} - R_n &= \|x + \Delta\| - \|x\| \\ &= [\ldots \text{expand using Taylor's theorem} \ldots] \\ &= \Delta_x + \frac{\|\Delta\|^2 - \Delta_x^2}{2\|x\|} + O(\|x\|^{-2}). \end{aligned}$$

So,

$$\mu_1(r) = \frac{(d-1)b^2}{d}\frac{1}{2r} + O(r^{-2}), \quad \mu_2(r) = \frac{a^2}{d} + O(r^{-1}).$$

Theorem

Let (X_n) be an elliptical random walk in $\mathbb{R}^d$, with parameters a and b.

- *If $(d-1)b^2 - a^2 > 0$ then (X_n) is transient.*
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Simulations

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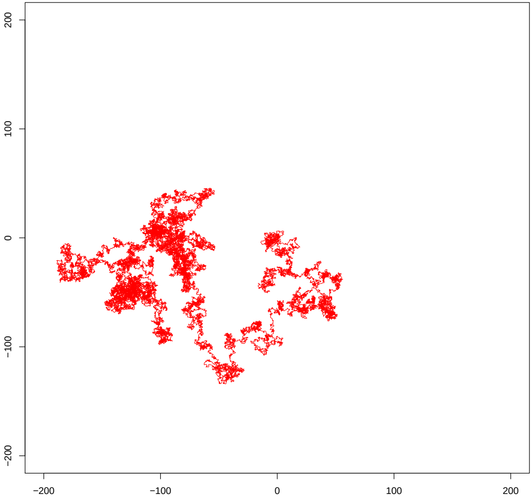
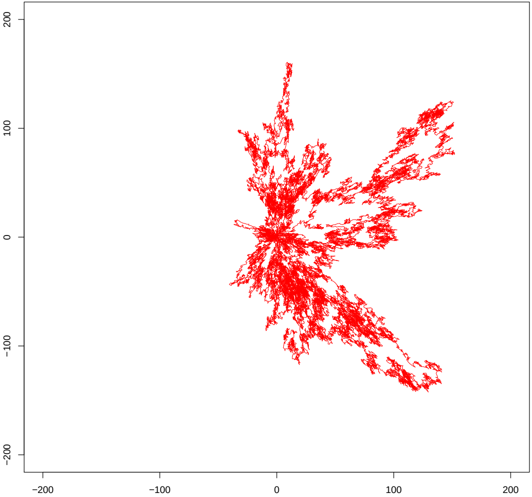


Simulations

$a = 2, b = 1$



Simulations

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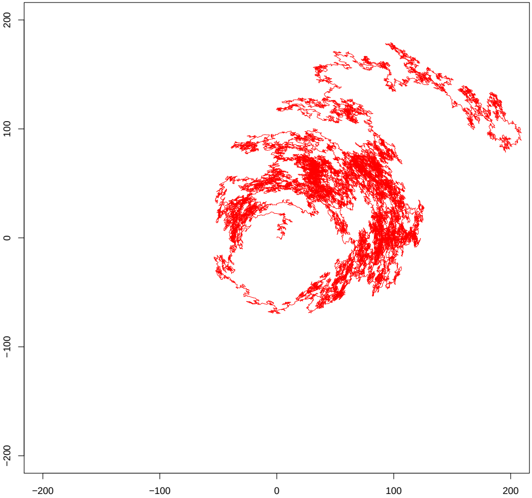
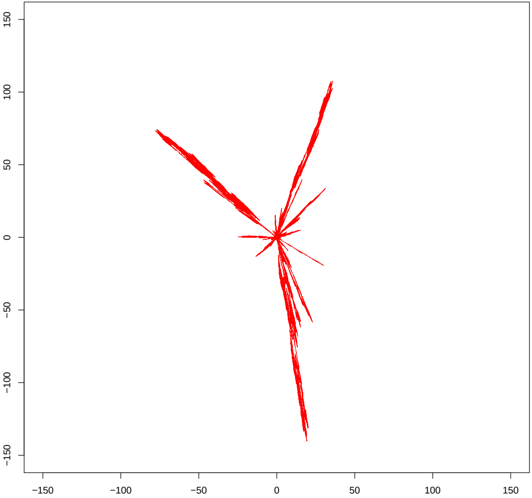


Simulations

$a = 1, b = 0.05$



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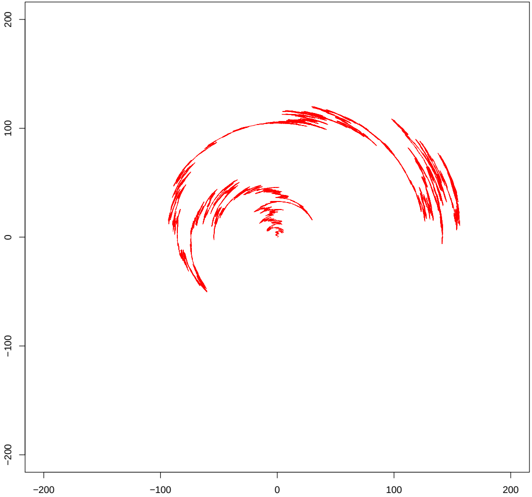

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Diffusion limits

Back to homogeneous case:

Theorem (Donsker)

The Pearson–Rayleigh walk in $\mathbb{R}^d$ *(the case* $a \equiv b = 1$*) converges to* d*-dimensional Brownian motion:*

$$\left(\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \right)_{t \in [0,1]} \Longrightarrow (b_t)_{t \in [0,1]} \, .$$

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Now, more generally:

Theorem

If (X_n) is an elliptical random walk in $\mathbb{R}^d$, then there exists a continuous strong Markov process (a diffusion) $(\mathcal{X}_t)$ on $\mathbb{R}^d$, whose law depends on the parameters a and b, such that,

$$\left(\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}} \right)_{t \in [0,1]} \Longrightarrow (\mathcal{X}_t)_{t \in [0,1]} .$$

Brownian motion and Bessel processes

Brownian motion on $\mathbb{R}^d$ ($d \geq 2$) possesses a skew-product representation.

$$\text{Let} \quad r_t := \|b_t\|, \quad \theta_t := \frac{b_t}{\|b_t\|}. \quad \text{Then,}$$

- r_t is a Bessel process on $\mathbb{R}_+$ of 'dimension' (parameter) d;
- θ_t is a (stochastic) time-change of an independent Brownian motion on the sphere.

- A Bessel process with 'dimension' δ, BES(δ), is a Markov process β_t on $\mathbb{R}_+$ satisfying the SDE

$$\mathrm{d}\beta_t = \frac{\delta - 1}{2\beta_t} \mathbf{1}_{\{\beta_t \neq 0\}} \mathrm{d}t + \mathrm{d}W_t,$$

  where W_t is BM on $\mathbb{R}$.
- $0 \in \mathbb{R}_+$ is recurrent for BES(δ) if $1 \leq \delta < 2$ and transient if $\delta \geq 2$.

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- r_t is a Bessel process on $\mathbb{R}_+$ of 'dimension' (parameter) d;
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- Define the additive functional $\rho(t) := \int_0^t r_s^{-2} \mathrm{d}s$.
- Then $\theta_t = \varphi_{\rho(t)}$, where φ_t is BM on $\mathbb{S}^{d-1}$ independent of r_t.
- That is, φ_t solves the SDE

$$\mathrm{d}\varphi_t = -\frac{d-1}{2}\varphi_t \mathrm{d}t + (I - \varphi_t \varphi_t^\top)\mathrm{d}W_t,$$

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- θ_t is a time-change of a two-sided BM $(\varphi_t)_{t\in\mathbb{R}}$ on $\mathbb{S}^{d-1}$, independent of r_t.

General setting for invariance principle

- Moments condition: $\sup_x \mathbb{E}_x[\|\Delta\|^4] < \infty$.
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- Asymptotic isotropy: $M(x) \to \sigma^2(\hat{x})$ as $\|x\| \to \infty$ for a positive-definite matrix valued $\mathcal{C}^\infty$-function σ^2 on $\mathbb{S}^{d-1}$.

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Define for each $u \in \mathbb{S}^{d-1}$ an inner product $\langle \cdot , \cdot \rangle_u$ on $\mathbb{R}^d$ via

$$\langle y, z \rangle_u := y^\top \cdot \sigma^2(u) \cdot z = \langle y, \sigma^2(u) \cdot z \rangle, \quad (\text{for } y, z \in \mathbb{R}^d).$$

- Limiting covariance regularity: There exist constants $U, V, \delta > 0$ such that, for all $u, v \in \mathbb{S}^{d-1}$,

$$\langle u, u \rangle_u = U, \quad \operatorname{tr} \sigma^2(u) = V, \quad \text{and} \quad \langle v, v \rangle_u \geq \delta.$$

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- Limiting radial structure: $u \in \mathbb{S}^{d-1}$ is eigenvector of $\sigma^2(u)$.

General setting for invariance principle

Theorem

If (X_n) is a random walk in $\mathbb{R}^d$ of the above type, then there exists a continuous strong Markov process (a diffusion) $(\mathcal{X}_t)$ on $\mathbb{R}^d$ such that,

$$\left(\frac{X_{\lfloor nt \rfloor}}{\sqrt{n}}\right)_{t\in[0,1]} \Longrightarrow (\mathcal{X}_t)_{t\in[0,1]}.$$

The diffusion $(\mathcal{X}_t)$ is the unique weak solution of the SDE

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Typically $x \mapsto \sigma(\hat{x})$ has a discontinuity at $0 \in \mathbb{R}^d$ *and* $(\mathcal{X}_t)$ keeps visiting 0, so standard methods from (Ethier & Kurtz, 1986) need to be extended (key fact: Bessel local time at 0 vanishes).

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The martingale problem

$$d\mathcal{X}_t = \sigma(\hat{\mathcal{X}}_t)dW_t, \qquad \textit{for any deterministic} \quad \mathcal{X}_0 \in \mathbb{R}^d,$$

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- Excursion theory for $(\mathcal{X}_t)$ has to be developed for *uniqueness in law* (works for any square-root σ).
- *Strong existence* and *pathwise uniqueness* may fail even for smooth σ (depends on the choice of square-root).

General setting: the excursion skew-decomposition

Let $(\xi_t, t \geq 0)$ be BM on $\mathbb{R}^d$. Then SDE

$$d\psi_t = (\sigma(\psi_t) - \psi_t \psi_t^\top) d\xi_t - \frac{V-1}{2} \psi_t dt, \qquad \psi_0 \in \mathbb{S}^{d-1}, \qquad (1)$$

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(b) *Skew-product structure. Let $s > 0$ and $\tau_s := \inf\{t \geq s : r_t = 0\}$. Then for any $t \in [s, \tau_s)$,*

$$\hat{\mathcal{X}}_t = \varphi_{\rho_s(t)}, \qquad \text{where } \rho_s(t) = \int_s^t r_u^{-2} du,$$

processes φ and r are independent and φ follows SDE (1) started according its stationary measure μ.

Properties of $\mathcal{X}$

Scaling: $\mathcal{X}$ and $\mathcal{Y} = (c^{-1/2}\mathcal{X}_{ct})$, $c > 0$, have the same law:
$d\mathcal{Y}_t = c^{-1/2} d\mathcal{X}_{ct} = c^{-1/2}\sigma(\widehat{\mathcal{X}}_{ct}) dW_{ct} = \sigma(\widehat{\mathcal{Y}}_t) d(c^{-1/2} W_{ct})$

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Rapid spinning: Let $s > 0$ and $\tau_s^- = \sup\{t < s : r_t = 0\}$. For any $t \in (\tau_s^-, \tau_s)$ in excursion interval, it holds

$$\lim_{s \downarrow \tau_s^-} \rho_s(t) = \infty, \qquad \text{where} \quad \rho_s(t) = \int_s^t r_u^{-2}\mathrm{d}u. \tag{2}$$

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Proof of (2): (Pitman & Yor, 1982) BES(U/V) excursion (recall $\delta = U/V \in (1,2)$): pick maximum according to σ-finite measure $m^{3-\delta}\mathrm{d}m$ and run back-to-back two independent BES($4-\delta$) from 0 it hits m. Apply (M & Urusov, 2012).

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$$\mathcal{G} = (1/2)\Delta_g + b,$$

where Δ_g is the Laplace–Beltrami operator on $(\mathbb{S}^{d-1}, g)$ and vector field b is explicit in σ^2 and the metric g. Here,

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$$b = \frac{1}{2} \left((d - V)x_i - \frac{\partial \sigma^2_{ij}}{\partial x_j} + \frac{1}{2} g^{ik} g_{j\ell} \frac{\partial \sigma^2_{j\ell}}{\partial x_k} \right) \frac{\partial}{\partial x_i}.$$

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$\exists!$ invariant measure μ on $\mathbb{S}^{d-1}$, such that $\mu(dx) = \nu(x)dx$.
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Hence excursion representation for BES(U/V) (in $\mathbb{R}_+$) from (Pitman & Yor, 1982) extends to $\mathcal{X}$ (in $\mathbb{R}^d$).

Some remarks

Walsh's Brownian motions: degenerate case $U = V$ is excluded from our results. But for U very close to V the measure on $\mathbb{S}^d$ from Walsh's construction is our stationary measure angular measure μ of ψ. Heuristically this approximates Walsh's Brownian motion (recall simulation).

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(i) pathwise uniqueness holds (implying strong uniqueness);

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Which of (i) or (ii) occurs **does** depend on the choice of square-root σ (e.g. multidimensional Tanaka SDE). $\exists$**smooth** σ under (ii) (including "complex Brownian motion" (Stroock & Yor, 1981). We have examples for $d = 2, 4, 8$.

Marginal limit theorem

At time $t = 1$, the law of $\mathcal{X}_1$ is given by

- $\|\mathcal{X}_1\|^2 \sim \Gamma(\frac{1}{2} + (d-1)\frac{b^2}{2a^2}, 2a^2)$ (Gamma);
- $\hat{\mathcal{X}}_1 \sim U(\mathbb{S}^{d-1})$ (uniform);
- $\|\mathcal{X}_1\|$ and $\hat{\mathcal{X}}_1$ are independent.

(When $a = b$ then $\|\mathcal{X}_1\|^2$ is a scalar multiple of a χ^2 random variable with d degrees of freedom.)

So for example we get an angular ergodic result for the random walk: for measurable $A \subseteq \mathbb{S}^{d-1}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbf{1}\{\hat{X}_k \in A\} = \frac{|A|}{|\mathbb{S}^{d-1}|}, \text{ in } L^1.$$

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Almost-sure version unlikely to hold as the limit is non-degenerate

$$\int_0^1 \mathbf{1}\{\hat{\mathcal{X}}_t \in \mathfrak{A}\}\mathrm{d}t = \lim_{\varepsilon\downarrow 0} \int_\varepsilon^1 \mathbf{1}\{\hat{\mathcal{X}}_t \in \mathfrak{A}\}\mathrm{d}t.$$

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