On exponential functionals of Lévy processes

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The Ornstein–Uhlenbeck process

- $\eta = (\eta_t)_{t \geq 0}$ Lévy process, $\lambda > 0$.
- Solution $V = (V_t)_{t \geq 0}$ to Langevin equation

$$dV_t = -\lambda V_t \, dt + d\eta_t,$$

  i.e.

$$V_t = V_0 - \int_0^t \lambda V_s \, ds + \eta_t, \quad t \geq 0,$$

  for some V_0, is called an Ornstein–Uhlenbeck process (driven by η).
- Solution well known to be

$$V_t = e^{-\lambda t} \left(V_0 + \int_0^t e^{\lambda s} \, d\eta_s \right), \quad t \geq 0$$

Theorem (Wolfe 1982, Sato and Yamazato 1984)

(a) The following are equivalent:

1. A starting random variable V_0 can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary, i.e. such that
$$(V_{t_1}, \ldots, V_{t_n}) \stackrel{d}{=} (V_{t_1+h}, \ldots, V_{t_n+h}) \quad \forall\, t_1, \ldots, t_n, h \geq 0.$$
2. The integral $\int_0^t e^{-\lambda s}\, d\eta_s$ converges almost surely to a finite random variable as $t \to \infty$.
3. $E \log^+ |\eta_1| < \infty$ (here, $\log^+(x) = \log(\max\{x, 1\})$).

(b) In that case, the stationary (marginal) distribution is unique and given by $\int_0^\infty e^{-\lambda t}\, d\eta_t$.
(c) Further, $\int_0^\infty e^{-\lambda t}\, d\eta_t$ is self-decomposable, and when varying η over all Lévy processes with finite log-moment, all self-decomposable distributions are obtained in this way.

Recall: A distribution $\mu = \mathcal{L}(X)$ is self-decomposable, if for each $c \in (0,1)$ there exists a random variable Y_c, independent of X, such that

$$cX + Y_c \stackrel{d}{=} X.$$

Each self-decomposable distribution is infinitely divisible, and the class of all self-decomposable distributions is denoted by $L(\mathbb{R})$.

The previous theorem implies that

$$\Phi : \{\mathcal{L}(\eta_1) \text{ inf. div. with } E \log^+ |\eta_1| < \infty\} \to L(\mathbb{R}),$$

$$\mathcal{L}(\eta_1) \mapsto \Phi(\mathcal{L}(\eta_1)) := \mathcal{L}\left(\int_0^\infty e^{-\lambda t}\, d\eta_t\right),$$

is a bijection. [It is even an algebraic isomorphism, i.e. additionally respects convolution, see Jurek and Mason 1993].

Generalised Ornstein–Uhlenbeck process

- $(\xi, \eta) = (\xi_t, \eta_t)_{t \geq 0}$ bivariate Lévy process
- For each starting random variable V_0, the process $V = (V_t)_{t \geq 0}$ defined by

$$V_t = e^{-\xi_t}(V_0 + \int_0^t e^{\xi_{s-}} \, d\eta_s)$$

  is called a generalised Ornstein–Uhlenbeck (GOU) process.
- Introduced by de Haan and Karandikar (1989) as natural continuous time analogue of random recurrence equations, then investigated in more detail by Carmona, Petit and Yor (1997) and others.
- Throughout this talk assume that $\xi_t \to \infty$ a.s. as $t \to \infty$ and that ξ and η are independent!

Theorem

1. A random variable V_0 can be chosen such that $(V_t)_{t\geq 0}$ is strictly stationary if and only if

$$V_\infty := \int_0^\infty e^{-\xi_{s-}}\, d\eta_s = \lim_{t\to\infty} \int_0^t e^{-\xi_{s-}}\, d\eta_s$$

   converges almost surely. Then $\mathcal{L}(V_\infty) = \mathcal{L}(V_0)$ is the strictly stationary marginal distribution.
   (Carmona, Petit, Yor 1997, L. and Maller 2005). We call V_∞ the exponential functional of (ξ, η).
2. Erickson and Maller (2005) characterise in terms of the Lévy triplet of (ξ, η) when $\int_0^\infty e^{-\xi_{s-}}\, d\eta_s$ converges almost surely. In particular, if $E\xi_1$ exists (then in $(0, \infty)$), then convergence if and only if $E \log^+ |\eta_1| < \infty$.

3. $(V_t)_{t\geq 0}$ is the unique solution of the SDE

$$dV_t = V_{t-}\, dU_t + d\eta_t, \quad t \geq 0,$$

where U a Lévy process such that

$$\mathcal{E}(U)_t = e^{-\xi_t} \quad \text{(stochastic exponential)},$$

i.e.

$$U_t = -\xi_t + \sum_{0<s\leq t} \left(e^{-\Delta \xi_s} - 1 + \Delta \xi_s) \right) + t\sigma_\xi^2/2, \quad t \geq 0$$

(by inverting the Doleans-Dade formula). The Lévy measure ν_U of U satisfies

$$\nu_U((-\infty, -1]) = 0$$

(e.g. Behme, L. and Maller, 2011).

Remark

- To determine the distribution of $\int_0^\infty e^{-\xi_{s-}}\, d\eta_s$ and properties of it has attracted a lot of attention, in particular that of $\int_0^\infty e^{-\xi_{s-}}\, ds$, e.g. Bertoin, Carmona, Chaumont, Dufresne, Gjessing, Kuznetsov, Pardo, Patie, Paulsen, Petit, Rivero, Savov, Yor and others.
- However, the explicit distribution is known only in a few special cases.

Setting and questions:

$\xi = (\xi_t)_{t\geq 0}$ a Lévy process with $\xi_t \to \infty$ a.s. ($t \to \infty$). Denote

$$D_\xi := \left\{ \mathcal{L}(\eta_1) : (\eta_t)_{t\geq 0} \text{ 1-dim. LP, indep. of } \xi \text{ s.t. } \int_0^\infty e^{-\xi_{s-}} d\eta_s \text{ conv. a.s.} \right\}$$

$$\begin{aligned} \Phi_\xi : D_\xi &\to \text{set of probability distributions,} \\ \mathcal{L}(\eta_1) &\mapsto \mathcal{L}\left(\int_0^\infty e^{-\xi_{s-}}\, d\eta_s \right), \\ R_\xi &:= \Phi_\xi(D_\xi) = \text{Range}(\Phi_\xi). \end{aligned}$$

Questions:

1. Is Φ_ξ injective?
2. Is R_ξ closed?
3. Is Φ_ξ or its inverse (if existent) continuous?
4. Characterize R_ξ.

For injectivity derive equation for characteristic function via the generator of the GOU process.

Proposition:

1. The GOU process is a Feller process.
2. $C_c^2(\mathbb{R})$, $C_c^\infty(\mathbb{R})$ and
$$\{f \in C_0^2(\mathbb{R}) : \lim_{|x|\to\infty} \left(xf'(x)| + |x^2 f''(x)|\right) = 0\}$$
are cores.
3. The generator of V can be expressed explicitly in terms of the characteristic triplets of η and ξ.

(Behme and L. 2013; for 3. also Carmona, Petit and Yor (1997), Kuznetsov, Pardo and Savov (2012) and others).

Theorem (Behme and L., 2015)

- $(\gamma_\xi, \sigma_\xi^2, \nu_\xi)$ characteristic triplet of ξ
- $\mathcal{L}(\eta_1) \in D_\xi$, $\mu := \mathcal{L}(V_\infty) = \mathcal{L}\left(\int_0^\infty e^{-\xi_{s-}}\, d\eta_s\right)$
- $u \mapsto \Psi_\eta(u)$ the characteristic exponent of η (i.e. $e^{\Psi_\eta(u)} = Ee^{iu\eta_1}$)
- Let $h \in C_c^\infty(\mathbb{R})$ with $h(x) = 1$ for $|x| \leq 1$ and $h(x) = 0$ for $|x| \geq 2$
- For $n \in \mathbb{N}$ and $u \in \mathbb{R}$ define

$$f_{n,u}(x) := e^{iux} h\left(\frac{x}{n}\right).$$

- Then

$$\begin{aligned}
&\Psi_\eta(u)\,\widehat{\mu}(u)\\
= \;& \lim_{n\to\infty}\Big[\gamma_\xi \int_{\mathbb{R}} x f'_{n,u}(x)\,\mu(dx) - \frac{\sigma_\xi^2}{2}\int_{\mathbb{R}} \Big(x^2 f''_{n,u}(x) + x f'_{u,n}(x)\Big)\,\mu(dx)\\
&- \int_{\mathbb{R}}\int_{\mathbb{R}} \left(f_{n,u}(xe^{-y}) - f_{n,u}(x) + xyf'_{n,u}(x)\mathbf{1}_{|y|\le 1}\right)\nu_\xi(dy)\,\mu(dx)\Big]
\end{aligned}$$

- If additionally $EV_\infty^2 = \int_{\mathbb{R}} x^2\,\mu(dx) < \infty$, then

$$\begin{aligned}
\Psi_\eta(u)\widehat{\mu}(u) \;=\;& \gamma_\xi\, u\,\widehat{\mu}'(u) - \frac{\sigma_\xi^2}{2}\Big(u^2\widehat{\mu}''(u) + u\,\widehat{\mu}'(u)\Big)\\
&- \int_{\mathbb{R}} \left(\widehat{\mu}(ue^{-y}) - \widehat{\mu}(u) + uy\widehat{\mu}'(u)\mathbf{1}_{|y|\le 1}\right)\nu_\xi(dy)
\end{aligned}$$

- Observe that right-hand sides completely determined by $\mu = \mathcal{L}(V_\infty)$ and $\mathcal{L}(\xi_1)$.

- $$\Psi_\eta(u)\widehat{\mu}(u) = \text{known function of } \widehat{\mu} \text{ and } \mathcal{L}(\xi_1)$$

  Hence when $\widehat{\mu}(u) \neq 0$ for u from a dense subset of $\mathbb{R}$, or if $\mathcal{L}(\eta_1)$ is already determined by its characteristic function $u \mapsto e^{\Psi_\eta(u)}$ in small neighbourhoods of zero, then Ψ_η and hence $\mathcal{L}(\eta_1)$ can be recovered from $\widehat{\mu}$ and $\mathcal{L}(\xi_1)$.

Remark:

- This can be used to derive sufficient conditions for injectivity.
- When η is a subordinator, a similar equation can be derived using Laplace transforms.

Theorem (Behme, L., 2015):

1. If ξ is spectrally negative, or if $\xi = qN$ for some $q > 0$ and a Poisson process N, then $\mu = \mathcal{L}(\int_0^\infty e^{-\xi_{s-}}\, d\eta_s)$ is self-decomposable (due to Samorodnitsky) resp. infinitely divisible, hence $\widehat{\mu}(u) \neq 0$ for all u, hence Φ_ξ is injective.
2. If η is symmetric, then $\widehat{\mu}(u) > 0$ for all $u \in \mathbb{R}$, hence $\mathcal{L}(\eta_1)$ is determined by μ and $\mathcal{L}(\xi_1)$.
3. If ν_η has some one-sided exponential moment, i.e. if

   $$\int_1^\infty e^{\varepsilon x}\, \nu_\eta(dx) < \infty \quad \text{or} \quad \int_{-\infty}^{-1} e^{-\epsilon x}\, \nu_\eta(dx) < \infty \quad \text{for some } \varepsilon > 0,$$

   then $\mathcal{L}(\eta_1)$ is determined by its values in small neighbourhoods of 0, hence $\mathcal{L}(\eta_1)$ is determined by μ and $\mathcal{L}(\xi_1)$.

Remark:

- Similarly, it is possible to express

$$\Psi_{-\xi}(u)\, Ee^{iu \log|V_\infty|}$$

as a function of $\mu = \mathcal{L}(\int_0^\infty e^{-\xi_{s-}}\, d\eta_s)$ and $\mathcal{L}(\eta_1)$.

- In particular, when

$$Ee^{-\varepsilon \log|V_\infty|} = E|V_\infty|^{-\varepsilon} < \infty \quad \text{for some } \varepsilon > 0,$$

then $\mathcal{L}(\xi_1)$ is determined by μ and $\mathcal{L}(\eta_1)$. This is the case for example when $\sigma_\eta^2 > 0$ (non-trivial Brownian motion part), or when η is a subordinator with strictly positive drift, giving injectivity of the mappping

$$\mathcal{L}(\eta_1) \mapsto \mathcal{L}\left(\int_0^\infty e^{-\xi_{s-}}\, d\eta_s\right)$$

in these cases.

Recall that

$$\Psi_\eta(u)\widehat{\mu}(u) = \gamma_\xi\, u\,\widehat{\mu}'(u) - \frac{\sigma_\xi^2}{2}\left(u^2\widehat{\mu}''(u) + u\,\widehat{\mu}'(u)\right) - \int_{\mathbb{R}} \left(\widehat{\mu}(ue^{-y}) - \widehat{\mu}(u) + uy\widehat{\mu}'(u)\mathbf{1}_{|y|\le 1}\right)\nu_\xi(dy)$$

when $EV_\infty^2 < \infty$. This can be used to characterise when $N(0,1)$ can be represented as $\mathcal{L}(\int_0^\infty e^{-\xi_{s-}}\, d\eta_s)$.

Theorem (Behme, L., 2015)
Let ξ, η be two independent Lévy process such that $V_\infty = \int_0^\infty e^{-\xi_{s-}}\, d\eta_s$ converges almost surely. Then $\mathcal{L}(V_\infty) = N(0,1)$ if and only if $\xi_t = \gamma_\xi t$ and $\eta_t = (2\gamma_\xi)^{1/2} W_t$, where $(W_t)_{t\ge 0}$ is a standard Brownian motion.

Continuity and closedness

It can be shown that $\Phi_\xi : \mathcal{L}(\eta_1) \mapsto \mathcal{L}(\int_0^\infty e^{-\xi_{s-}}\, d\eta_s$ is not continuous with respect to weak convergence (even when $\xi_t = t$). However:

Theorem (Behme, L., Maejima, 2016+)

- $R_\xi = \Phi_\xi(D_\xi)$ is closed under weak convergence.
- If Φ_ξ is injective, then Φ_ξ^{-1} is continuous with respect to weak convergence.
- If Φ_ξ^+ denotes Φ_ξ restricted to all non-negative laws $\mathcal{L}(\eta_1) \in D_\xi$, then Φ_ξ^+ is injective and $(\Phi_\xi^+)^{-1}$ is continuous.
- R_ξ is in general not closed under convolution.

Proof:

- Uses that Φ_ξ is closed mapping in the sense that if

$$\mathcal{L}(\eta_1^{(n)}) \in D_\xi, \quad \eta_1^{(n)} \stackrel{d}{\to} \eta_1 \quad \text{and} \quad \Phi_\xi(\mathcal{L}(\eta_1^{(n)})) \stackrel{w}{\to} \mu$$

  as $n \to \infty$, then

$$\mathcal{L}(\eta_1) \in D_\xi \quad \text{and} \quad \Phi_\xi(\mathcal{L}(\eta_1)) = \mu.$$

- Further, shows that $(\eta_1^{(n)})_{n\in\mathbb{N}}$ is tight if $(\Phi_\xi(\mathcal{L}(\eta_1^{(n)}))_{n\in\mathbb{N}}$ is tight.
- To show that it is in general not closed under convolution (e.g. if $Ee^{-2\xi_1} < 1$ and ξ not deterministic), use that there exists some symmetric $V = \int_0^\infty e^{-\xi_{s-}}\, d\eta_s \in R_\xi$ with finite variance. If R_ξ were closed under convolution, then by the central limit theorem and closedness of R_ξ, also $N(0,1)$ would be in R_ξ, a contradiction. $\square$

Characterisation of positive distributions in the range

Theorem (Behme, L., Maejima, 2016+)

Let ξ drift to $+\infty$ and $\mu = \mathcal{L}(V)$ be a probability distribution on $[0,\infty)$ with Laplace exponent ψ_V, i.e. $Ee^{-uV} = e^{\psi_V(u)}$, $u \geq 0$. Then the following are equivalent:

- $\mu \in R_\xi$.
- The function $g_\mu : (0,\infty) \to \mathbb{R}$, defined by

$$g_\mu(u) := (\gamma_\xi - \frac{\sigma_\xi^2}{2}) u \psi_V'(u) - \frac{\sigma_\xi^2}{2} u^2 \left(\psi_V''(u) + (\psi_V'(u))^2 \right) - \int_{\mathbb{R}} \left(e^{\psi_V(ue^{-y}) - \psi_V(u)} - 1 + u\psi_V'(u) y \mathbf{1}_{|y|\leq 1} \right) \nu_\xi(dy),$$

  defines the Laplace exponent of some subordinator η.

In that case,

$$\Phi_\xi(\mathcal{L}(\eta_1)) = \mu.$$

Specialisation to $\xi_t = \sigma B_t + at$

Corollary (Positive stable distributions in the range)

- $\xi_t = \sigma B_t + at$, $\sigma, a > 0$, B Standard BM
- μ a positive α-stable distribution with Lévy density $x \mapsto cx^{-1-\alpha}$ on $(0, \infty)$
- Then $\mu \in R_\xi$ if and only the drift of μ is zero and $\alpha \in (0, \frac{2a}{\sigma^2} \wedge \frac{1}{2}]$
- In that case, if $\alpha < 1/2$, then $\mu = \Phi_\xi(\mathcal{L}(\eta_1))$, where η is a subordinator with drift 0 and Lévy density on $(0, \infty)$ given by

$$x \mapsto c\alpha \left(a - \frac{\sigma^2}{2} \right) x^{-\alpha-1} + \sigma^2 c^2 \frac{\alpha(\Gamma(1-\alpha))^2}{\Gamma(1-2\alpha)} x^{-2\alpha-1}$$

- If $\alpha = 1/2 = 2a/\sigma^2$, then $\mu = \Phi_\xi(\mathcal{L}(\eta_1))$, where η is a deterministic subordinator with drift $\sigma^2 c^2((\Gamma(1-\alpha))^2/2$.

Corollary:

Let $\xi_t = \sigma B_t + at$, $\sigma, a > 0$.

- Then R_ξ contains the closure of all finite convolutions of positive α-stable distributions with drift 0 and $\alpha \in (0, \frac{2a}{\sigma^2} \wedge \frac{1}{2}]$.
- If $\mu \geq 0$ is in R_ξ, then μ must have drift 0.

Corollary (Nested ranges)

Let $\xi^{(a,\sigma)} = \sigma B_t + at$ and consider $R^+_{\xi^{(a,\sigma)}}$, the set of positive distributions in $R_{\xi^{(a,\sigma)}}$. Then

$$
\begin{aligned}
R^+_{\xi^{(a,\sigma)}} &= R^+_{\xi^{(a/\sqrt{\sigma},1)}}, \\
R^+_{\xi^{(a,\sigma)}} &\subset R^+_{\xi^{(a',\sigma)}} \quad \text{for} \quad a' \geq a, \\
R^+_{\xi^{(a,\sigma)}} &\supset R^+_{\xi^{(a,\sigma')}} \quad \text{for} \quad \sigma' \geq \sigma, \\
\bigcup_{a,\sigma>0} R^+_{\xi^{(a,\sigma)}} &\subsetneq L(\mathbb{R}^+).
\end{aligned}
$$

Remark: Extensions of some of the above results to Lévy processes with jumps can be found in Behme (2015).

References

Talk based on:

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