Spines in growth-fragmentation models

Alex Watson
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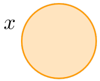
Lévy 8 | Angers
29 July 2016

A finite-activity model



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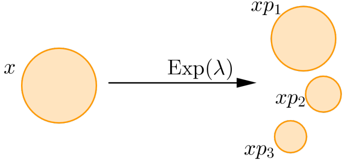
$K(\mathrm{d}\mathbf{p})$



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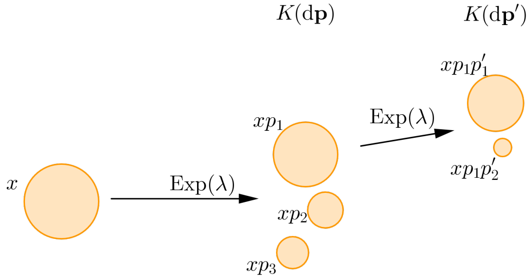

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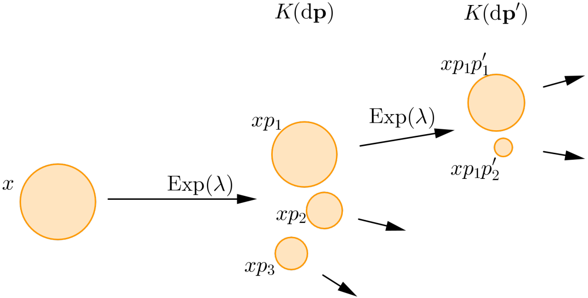

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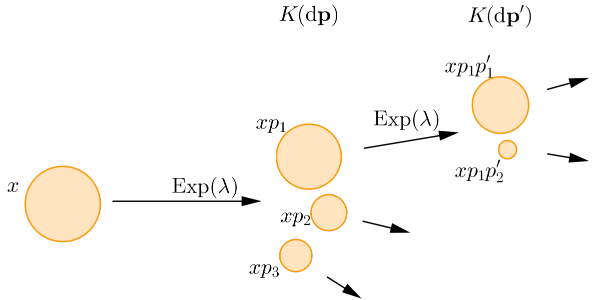

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  $\mathcal{P} = \{(p_1, p_2, \dots) : p_1 \geq p_2 \geq \cdots, \sum_{i\geq 1} p_i \leq 1\}$.

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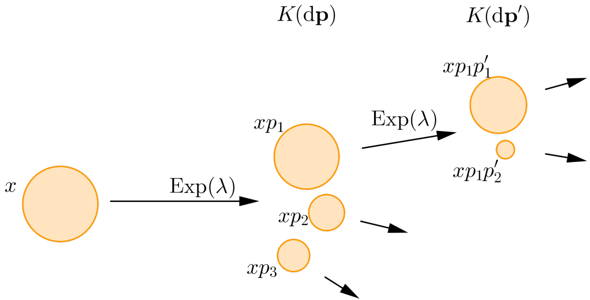

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- Mean measure: $\langle \mu_t, f \rangle = \int f(x)\, \mu_t(\mathrm{d}x) = \mathbb{E}\Big[\sum_{u \text{ particle alive at } t} f(\mathrm{size}(u))\Big]$

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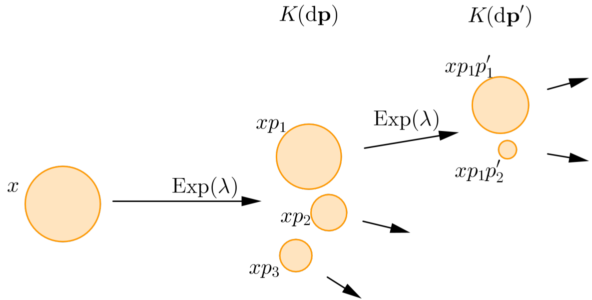

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- $\nu(\mathrm{d}\mathbf{p}) = \lambda K(\mathrm{d}\mathbf{p})$

The pure fragmentation equation

$$\partial_t \langle \mu_t, f \rangle = \left\langle \mu_t, \int_{\mathcal{P}} \Big\{ \sum_{i \geq 1} f(xp_i) - f(x) \Big\} \, \nu(\mathrm{d}\mathbf{p}) \right\rangle, \qquad f \in C_c^\infty(0,\infty),$$

$$\mu_0 = \delta_1$$

The growth-fragmentation equation

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- $a \in \mathbb{R}$
- Require only $\int (1 - p_1)^2 \, \nu(\mathrm{d}\mathbf{p}) < \infty$ (asymmetric children)

Questions for today

- Existence and representation
- Explore many-to-one theorem

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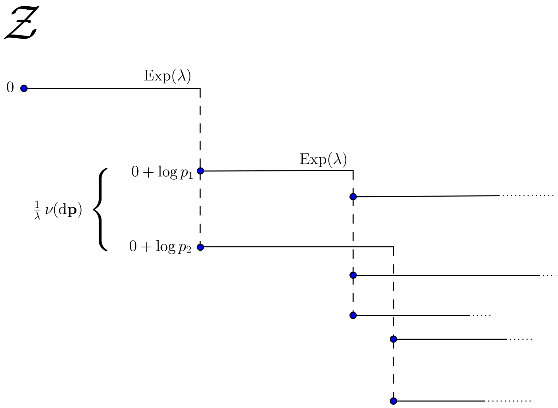
Fragmentation processes ($\nu < \infty$)

$$\mathcal{Z}(t) = \sum_{u \text{ fragments}} \delta_{\log(\mathrm{size}(u))} \mathbb{1}_{\{u \text{ alive at time } t\}}$$



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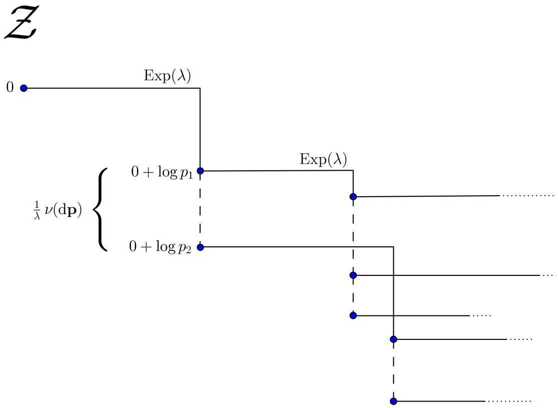
This is a compound Poisson process with immigration.



Compensated fragmentation processes, $\int (1-p_1)^2 \, \nu(\mathrm{d}\mathbf{p}) < \infty$

- We can build $\mathcal{Z}$ in general
- Create a Lévy process whose Lévy measure is $\nu(\log p_1 \in \cdot)$
- At every jump of size z, sample from $\nu(\mathrm{d}\mathbf{p} \mid \log p_1 = z)$ and immigrate new particles at relative positions $\log p_i$, $i \geq 2$.

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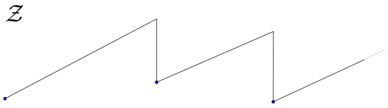

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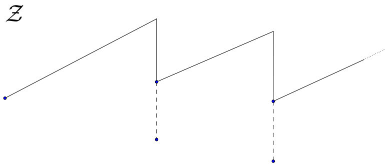

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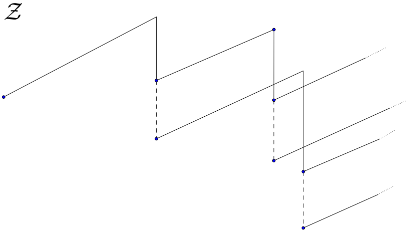

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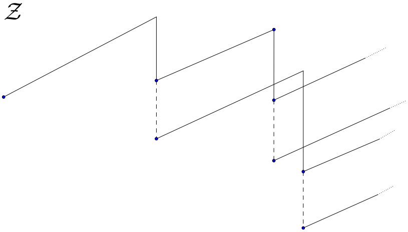

(Aside: let's assume we have a consistent way to give the particles labels $u \in \mathcal{U}$. Write $\mathcal{Z}_u(t)$ for the position of particle u at time t.)

Solution of the equation

Define the cumulant κ: $\mathbb{E}\big[\sum_u e^{q\mathcal{Z}_u(t)}\big] = e^{t\kappa(q)}$. It satisfies

$$\kappa(q) = aq + \int_{\mathcal{P}} \Big\{ \sum_{i\geq 1} p_i^q - 1 + (1-p_1)q \Big\}\, \nu(\mathrm{d}\mathbf{p}).$$

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Fix ω such that $\kappa(\omega) < \infty$.

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- *Let*

$$\langle \mu_t, f \rangle = e^{t\kappa(\omega)} \mathbb{E}\big[e^{-\omega\xi(t)} f\big(e^{\xi(t)}\big)\big] = \mathbb{E}\Big[\sum_u f\big(e^{\mathcal{Z}_u(t)}\big)\Big].$$

This is the unique solution of the growth-fragmentation equation.

A martingale for the fragmentation

- Let

$$W(\omega, t) = e^{-t\kappa(\omega)} \sum_u e^{\omega \mathcal{Z}_u(t)}, \qquad t \geq 0.$$

The process W is the additive martingale.

A martingale for the fragmentation

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The process W is the additive martingale.
- It induces new measure $\mathbb{P}_\omega$ for $\mathcal{Z}$: if F_t is measurable with respect to paths up to time t,
$$\mathbb{E}_\omega[F_t(\mathcal{Z})] = \mathbb{E}[F_t(\mathcal{Z}) W(\omega, t)].$$

Construction of a spine process, I

Let ξ be the Lévy process with Laplace exponent $\kappa(\omega + \cdot) - \kappa(\omega)$.

- Lévy measure: $\Pi(\mathrm{d}z) = \sum_{i \geq 1} e^{\omega z} \nu(\log p_i \in \mathrm{d}z)$.

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- Lévy measure: $\Pi(\mathrm{d}z) = \sum_{i \geq 1} e^{\omega z} \nu(\log p_i \in \mathrm{d}z)$.
- Jump process: $M(\mathrm{d}s, \mathrm{d}z)$, a Poisson point process with intensity measure $\mathrm{d}s \, \Pi(\mathrm{d}z)$.

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We will define a *decoration* of the jump process M:

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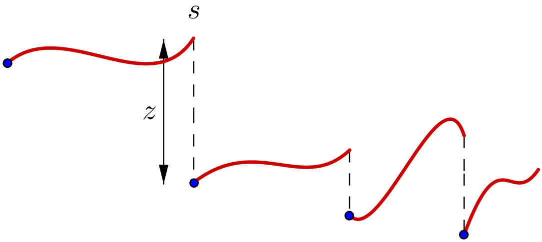

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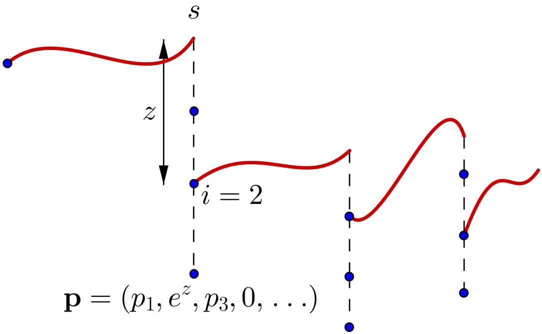

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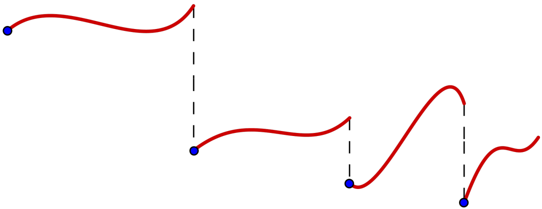

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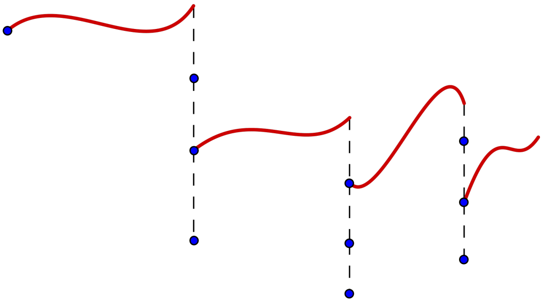

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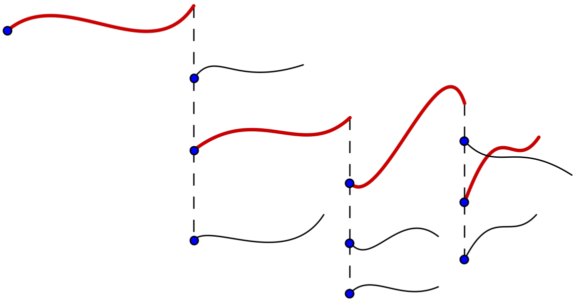

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We distinguish within $\tilde{\mathcal{Z}}$ the particle with position ξ: let U_t be such that $\tilde{\mathcal{Z}}_{U_t}(t) = \xi(t)$.

Many-to-one theorem

Theorem

For F_t measurable with respect to paths up to time t, and u any label,

$$\mathbb{E}_\omega[F_t(\tilde{\mathcal{Z}})\mathbb{1}_{\{U_t=u\}}] = e^{-t\kappa(\omega)}\mathbb{E}[F_t(\mathcal{Z})e^{\omega\mathcal{Z}_u(t)}].$$

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Corollary

Summing over u,

$$\mathbb{E}_\omega[F_t(\tilde{\mathcal{Z}})] = e^{-t\kappa(\omega)}\mathbb{E}[F_t(\mathcal{Z})W(\omega,t)] = \mathbb{E}_\omega[F_t(\mathcal{Z})].$$

That is, $\tilde{\mathcal{Z}} \overset{d}{=} \mathcal{Z}$ under $\mathbb{P}_\omega$.

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Corollary

For Borel f,

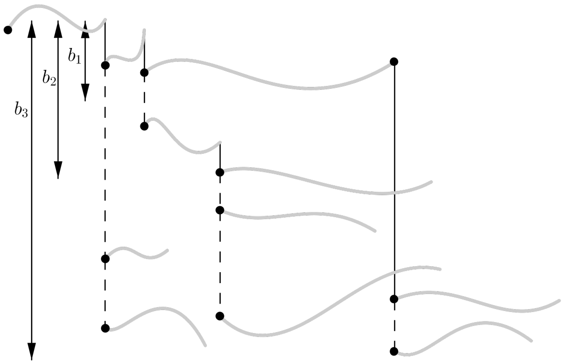
$$e^{t\kappa(\omega)}\mathbb{E}_\omega\left[e^{-\omega\tilde{\mathcal{Z}}_{U_t}(t)}f(\tilde{\mathcal{Z}}_{U_t}(t))\right] = \mathbb{E}\Big[\sum_u f(\mathcal{Z}_u(t))\Big].$$

Discussion of proof, I

We define *truncation* of the processes.

In the truncated process $\mathcal{Z}^{(b)}$, a child particle is kept only if:

- its displacement from the parent is less than $-b$, *or*
- it is the largest child



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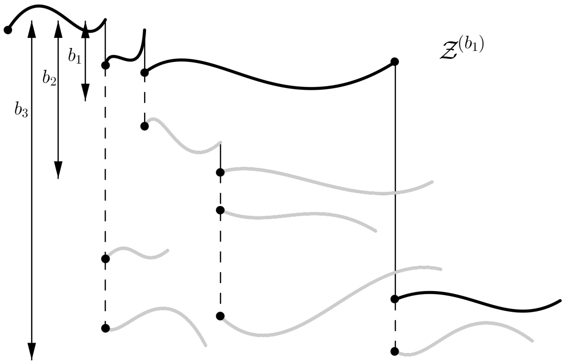

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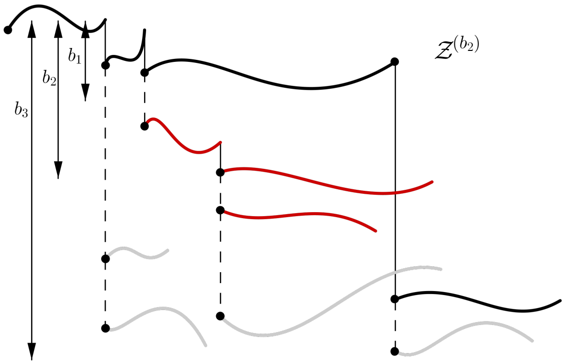

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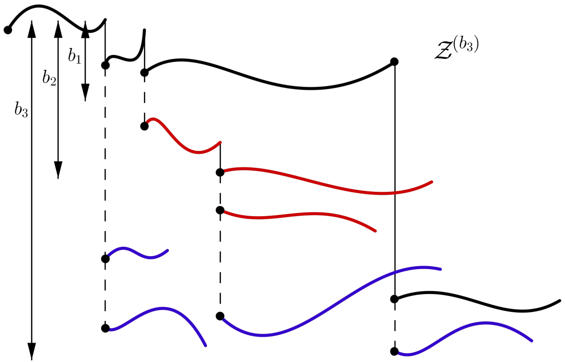

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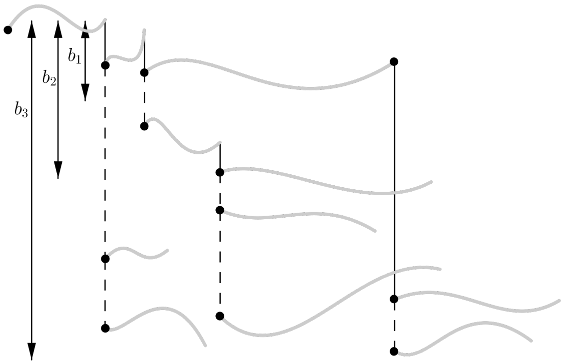

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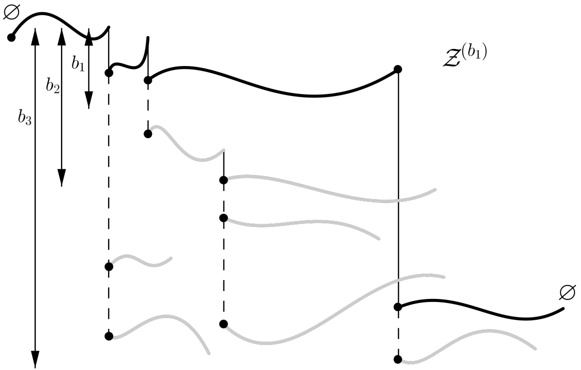

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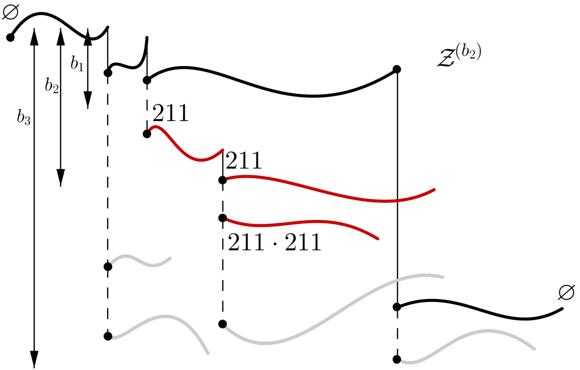

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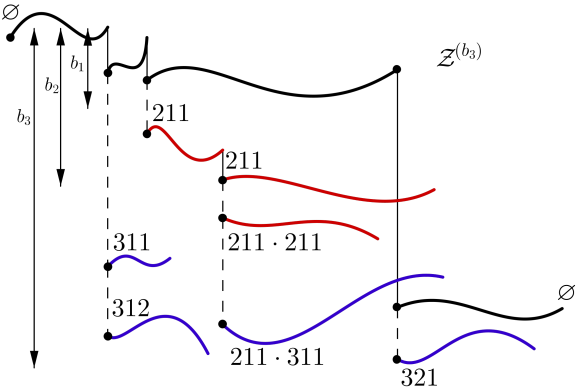

Discussion of proof, II

The truncation operation can be applied to:

- $\mathcal{Z}$, under $\mathbb{P}$ — let $\kappa^{(b)}$ be the cumulant of $\mathcal{Z}^{(b)}$
- $\tilde{\mathcal{Z}}$, under $\mathbb{P}_\omega$

They are related by:

Lemma

Let

$$\zeta = \inf\{t \geq 0 : U_t \text{ not in } \tilde{\mathcal{Z}}^{(b)}(t)\}.$$

Then,

$$\mathbb{E}_\omega\big[F_t(\tilde{\mathcal{Z}}^{(b)})\mathbb{1}_{\{U_t=u\}} \mid \zeta > t\big] = e^{-t\kappa^{(b)}(\omega)}\mathbb{E}\big[F_t(\mathcal{Z}^{(b)})e^{\omega \mathcal{Z}_u^{(b)}(t)}\big]$$

Further questions

- Derivative martingale
  $\partial W(\omega, t) = e^{-t\kappa(\omega)} \sum_u (\omega \mathcal{Z}_u(t) - t\kappa'(\omega)) e^{\omega \mathcal{Z}_u(t)}$
- KPP equation
- 'Non-homogeneous' fragmentations (self-similar done by Bertoin–Budd–Curien–Kortchemski '16)

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J. Bertoin
Compensated fragmentation processes and limits of dilated fragmentations

J. Bertoin, A. R. Watson
Probabilistic aspects of critical growth-fragmentation equations

A. R. Watson, Q. Shi
Tilting of compensated fragmentations [in preparation]

Thank you!