Power variation for a class of Lévy driven moving averages

Andreas Basse-O'Connor
joint work with R. Lachièze-Rey and M. Podolskij

Aarhus University, Denmark

The 8th International Conference on Lévy Processes.
July 25–29, 2016. Angers, France.

$\ldots \leftarrow$ **Lévy processes** $\rightarrow \ldots$

$\swarrow \quad \searrow$

Stochastic differential equations

Markov processes

Semimartingales

Gaussian processes or

Infinitely divisible processes

1. A random vector X is called infinitely divisible if for all $n \geq 1$ there exists $Y_1, \ldots, Y_n$ i.i.d. such that

$$X \stackrel{\mathcal{D}}{=} Y_1 + \cdots + Y_n.$$

2. A process $(X_t)_{t \in T}$ is called infinitely divisible if for all $n \geq 1$ and $t_1, \ldots, t_n \in T$, $(X_{t_1}, \ldots, X_{t_n})$ are infinitely divisible.

3. A Lévy process is an example of an infinitely divisible process.

4. Typically, infinitely divisible processes are:
   1. not Markov processes
   2. not semimartingales
   3. do not have independent increments

A key class of stationary infinitely divisible processes are the *moving averages*

$$X_t = \int_{\mathbb{R}} g(t-s)\, dL_s$$

1. $g : \mathbb{R} \to \mathbb{R}$ is a deterministic function
2. $L = (L_t)_{t\in\mathbb{R}}$ is a Lévy process indexed by $\mathbb{R}$.

Assumptions:

1. $$X_t = \int_{\mathbb{R}} \{g(t-s) - g_0(-s)\}\, dL_s$$
2. L is a symmetric Lévy process $\sim (0, \sigma^2, \nu)$
3. $g(t) \sim c_0 t^\alpha$ as $t \to 0$, $\alpha > 0$
4. $g \in C^1((0,\infty))$

Remark: (X_t) is an infinitely divisible process with stationary increments. Moreover, X has typical continuous sample paths!

The Blumenthal-Getoor index β of $L = (L_t)_{t \in \mathbb{R}}$ is defined as

$$\beta := \inf \left\{ r \geq 0 : \int_{-1}^{1} |x|^r \, \nu(dx) < \infty \right\}.$$

Fractional Lévy processes

- In the special case $g(t) = g_0(t) = t_+^\alpha$, X is called a *fractional Lévy process* and has the form

$$X_t = \int_{-\infty}^{t} \{(t-s)^\alpha - (-s)_+^\alpha\}\, dL_s.$$

- If in addition, L is an β-stable Lévy process then X is the *linear fractional stable motion* with Hurst index $H = \alpha + 1/\beta$. Here X is self-similar with index H, i.e. for all $a > 0$

$$(X_{at})_{t\geq 0} \overset{\mathcal{D}}{=} (a^H X_t)_{t\geq 0}.$$

For $\beta = 2$, X is the *fractional Brownian motion* is Hurst index $H := \alpha + 1/2$.

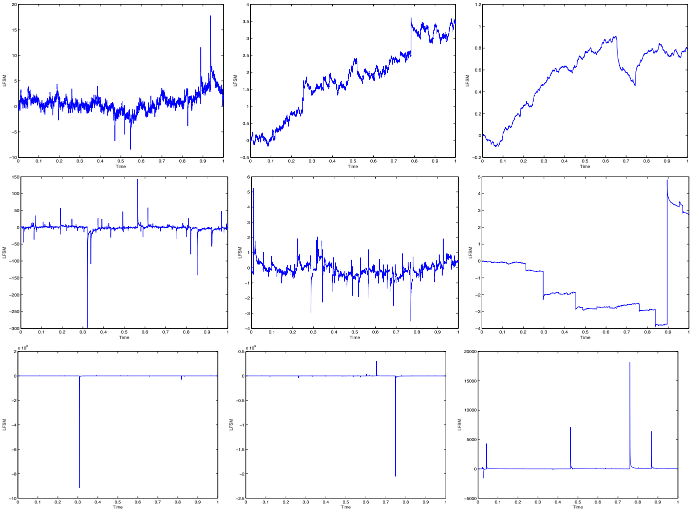

FIGURE 1. Top, middle and bottom panels: realizations of linear fractional stable motions for $\alpha = 1.8$, $\alpha = 1.2$ and $\alpha = 0.6$. In all cases, the left panel corresponds to $H = .2$, the middle panel to $H = .5$ and the right panel to $H = .8$. The $x-$axis represents time ($t = k/n,\ k = 0, 1, 2, \cdots, n$), while on the $y-$axis the values of the LFSM process are given.

Power variations

- For a stochastic process $X = (X_t)_{t \geq 0}$ and $p > 0$ we define the the power variation of X by

$$V(p)_n := \sum_{i=1}^{n} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^p.$$

In the following we will study the asymptotic behaviour of the functional $V(p)_n$ as $n \to \infty$.

Very little is known outside the two settings:

1. Itô semimartingales
2. Gaussian processes.

Two exceptions are the two works

1. The work [1] on the quadratic variation of the Rosenblatt process.
2. The work [2] on power variation of a class of fractional Lévy processes.

---

[1] C. Tudor and F. Viens (2009). Variations and estimators for self-similarity parameters via Malliavin calculus. *Ann. Probab.* 37.

[2] A. Benassi, S. Cohen and J. Istas (2004). On roughness indices for fractional fields. *Bernoulli* 10(2), 357–373.

Power variation for the fractional Brownian motion: First order asymptotics

Let X be a fractional Brownian motion with Hurst exponent H.

Using ergodic theory it follows that:
First order asymptotics for X: For any $H \in (0,1)$ we have

$$n^{-1+pH} V(p)_n \xrightarrow{\mathbb{P}} m_p := \mathbb{E}[|X_1|^p] \qquad n \to \infty.$$

- We will see that the limit theory for power variation

$$V(p)_n = \sum_{i=k}^{n} |X_{\frac{i}{n}} - X_{\frac{i-1}{n}}|^p \qquad \text{as } n \to \infty$$

depends heavily on the interplay between the three parameters

$$\underbrace{p}_{\text{power}} \qquad \underbrace{\alpha}_{\text{behaviour of } g \text{ at } 0} \qquad \text{and} \qquad \underbrace{\beta}_{\text{BG-index of } L}$$

First order asymptotics for power variation

Theorem (B., Lachièze-Rey and Podolskij)

(i): Assume that L is a $S\beta S$ process with $\beta \in (0,2)$.
If $\alpha \in (0, 1 - 1/\beta)$ and $p < \beta$, we obtain

$$n^{p(\alpha+1/\beta)-1} V(p)_n \xrightarrow{\mathbb{P}} m_p.$$

First order asymptotics for power variation

Theorem (cont.)

Assume that $p \geq 1$.

(ii): If $\alpha > 1 - 1/p,\ p > \beta$ *or* $\alpha > 1 - 1/\beta,\ p < \beta$, *we deduce*

$$n^{p-1} V(p)_n \xrightarrow{\mathbb{P}} \int_0^1 |F_s|^p ds$$

with

$$F_s = \int_{-\infty}^{s} g'(s-u)\, dL_u.$$

Theorem (cont')

(iii): If $\alpha \in (0, 1 - 1/p)$ and $p > \beta$, we obtain

$$n^{\alpha p} V(p)_n \xrightarrow{\mathcal{L}-s} |c_0|^p \sum_{m:\ T_m \in [0,1]} |\Delta L_{T_m}|^p V_m$$

where $(T_m)_{m \geq 1}$ are jump times of L, $(V_m)_{m \geq 1}$ are certain i.i.d. sequence of random variables independent of L.

Theorem

(iii): If $\alpha \in (0, 1 - 1/p)$ and $p > \beta$, then

$$n^{\alpha p} V(p)_n \xrightarrow{\mathcal{L}-s} |c_0|^p \sum_{m:\, T_m \in [0,1]} |\Delta L_{T_m}|^p V_m := Z$$

1. The limit Z is infinitely divisible with Lévy measure

$$(\nu \otimes \eta) \circ \big((y, v) \mapsto |c_0 y|^p v\big)^{-1}$$

   where η denotes the law of

$$V = \sum_{l=0}^{\infty} |(l + U)^\alpha - (l + U - 1)_+^\alpha|^p,$$

   $U \sim \mathcal{U}[0, 1]$.

2. Convergence in probability does not hold.

Summary of first order asymptotics

Theorem

(i): Assume that L is a $S\beta S$ process with $\beta \in (0,2)$. If $\alpha \in (0, 1-1/\beta)$ and $p<\beta$, we obtain

$$n^{p(\alpha+1/\beta)-1} V(p)_n \xrightarrow{\mathbb{P}} m_p.$$

(ii): Assume $p \geq 1$. If $\alpha > k-1/p,\ p>\beta$ or $\alpha > k-1/\beta,\ p<\beta$, we deduce

$$n^{kp-1} V(p)_n \xrightarrow{\mathbb{P}} \int_0^1 |F_s^{(k)}|^p ds.$$

(iii): If $\alpha \in (0, k-1/p)$ and $p>\beta$, we obtain

$$n^{\alpha p} V(p)_n \xrightarrow{\mathcal{L}-s} |c_0|^p \sum_{m:\ T_m \in [0,1]} |\Delta L_{T_m}|^p V_m \sim ID.$$

Remarks

- The above three cases covers all possible cases besides the three boundary cases:

$$\alpha = k - 1/p, \qquad \alpha = k - 1/\beta, \qquad p = \beta.$$

Second order asymptotics for case (i)

"Classical" results of the form

$$a_n \sum_{i=1}^{n} Y_i \xrightarrow{d} U \qquad n \to \infty.$$

where $(Y_i)_{i\geq 1}$ is a stationary sequence which satisfies one of the following

1. $(Y_i)_{i\geq 1}$ are independent
2. $(Y_i)_{i\geq 1}$ are martingale difference
3. $(Y_i)_{i\geq 1}$ are Markov chain
4. $(Y_i)_{i\geq 1}$ are strongly mixing

are never applicable.

Second order asymptotics

Theorem (Breuer–Major [1], Taqqu [2])

Suppose that X is the fractional Brownian motion with Hurst index $H \in (0,1)$. The following assertions hold:

(i) *Assume that $H \in (0, 3/4)$. Then*

$$\sqrt{n}\left(n^{-1+pH}V(p)_n - m_p\right) \xrightarrow{d} \mathcal{N}(0, v_p).$$

(ii) *When $H \in (3/4, 1)$ it holds that*

$$n^{2-2H}\left(n^{-1+pH}V(p)_n - m_p\right) \xrightarrow{d} Z,$$

where Z is a Rosenblatt random variable.

[1] Breuer and Major (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *Journal of Multivariate Analysis* 13.

[2] Taqqu (1979). Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete* 50.

Second order asymptotics associated with case (i)

Theorem (B., Lachièze-Rey and Podolskij)

Assume that L is a $S\beta S$ process with $\beta \in (0,2)$.
For $\alpha \in (0, 1-1/\beta)$ and $p < \beta/2$, it holds that

$$n^{1-\frac{1}{(1-\alpha)\beta}} \left(n^{p(\alpha+1/\beta)-1} V(p)_n - m_p \right) \xrightarrow{d} S_{(1-\alpha)\beta}$$

where $S_{(1-\alpha)\beta}$ is a totally right skewed $(1-\alpha)\beta$-stable random variable with mean zero.