Stationary increments harmonizable stable fields: upper estimates on path behaviour

Antoine Ayache
and
Geoffrey Boutard

UMR CNRS 8524 Laboratoire Paul Painlevé
Université Lille 1

Organization of the talk

Stable random variables and stable stochastic integrals

Let Z be a real-valued random variable, and χ_Z its characteristic function defined as: $\forall\,\lambda \in \mathbb{R}$, $\chi_Z(\lambda) := \mathbb{E}(e^{i\lambda Z})$. Z is said to have a symmetric stable distribution of stability parameter $\alpha \in (0,2]$ and scale parameter $\sigma \in \mathbb{R}_+$, if:

$$\forall\,\lambda \in \mathbb{R}, \quad \chi_Z(\lambda) = \exp(-\sigma^\alpha |\lambda|^\alpha)\,. \tag{1.1}$$

$\rightarrow$ when $\alpha = 2$, Z reduces to a centered Gaussian random variable of variance $2\sigma^2$.
$\rightarrow$ The situation is very different when $\alpha \in (0,2)$ and $\sigma > 0$; the distribution of Z becomes heavy-tailed:

$$\mathbb{P}(|Z| > z) \sim c(\alpha)\sigma^\alpha z^{-\alpha}, \quad \text{when } z \to +\infty. \tag{1.2}$$

This, in particular, implies that:

$$\mathbb{E}(|Z|^\gamma) < +\infty \quad \text{when } \gamma < \alpha, \text{ and } \quad \mathbb{E}(|Z|^\gamma) = +\infty \quad \text{when } \gamma \geq \alpha. \tag{1.3}$$

We denote by $\widetilde{M}_\alpha$ a complex-valued rotationally invariant α-stable random measure on $\mathbb{R}^d$ with Lebesgue control measure.

The related stable stochastic integral is denoted by $\int_{\mathbb{R}^d} (\,\cdot\,)\,\mathrm{d}\widetilde{M}_\alpha$. It is a linear map on the Lebesgue space $L^\alpha(\mathbb{R}^d)$ such that, for any deterministic function $g \in L^\alpha(\mathbb{R}^d)$, the real part $\mathcal{R}e\{\int_{\mathbb{R}^d} g(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\}$ is a real-valued symmetric α-stable random variable with a scale parameter satisfying

$$\sigma\Big(\mathcal{R}e\Big\{\int_{\mathbb{R}^d} g(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\Big\}\Big)^\alpha = \int_{\mathbb{R}^d} \big|g(\xi)\big|^\alpha\, d\xi. \tag{1.4}$$

The equality (1.4) is reminiscent of the classical isometry property of Wiener integrals; in particular, it implies that $\mathcal{R}e\{\int_{\mathbb{R}^d} g_n(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\}$ converges to $\mathcal{R}e\{\int_{\mathbb{R}^d} g(\xi)\,\mathrm{d}\widetilde{M}_\alpha(\xi)\}$ in probability, when a sequence $(g_n)_n$ converges to g in $L^\alpha(\mathbb{R}^d)$. This will be useful for us.

A classical reference on stable distributions and related topics, including stable random measures and their associated stochastic integrals, is the book of Samorodnitsky and Taqqu (1994).

Stationary increments harmonizable stable fields

Let us now focus on the definition of $\left\{X(t), t \in \mathbb{R}^d\right\}$, the class of the real-valued stationary increments harmonizable stable fields we intend to study.

The main ingredient of the definiton is f, an arbitrary real-valued Lebesgue measurable even function on $\mathbb{R}^d$ satisfying the condition:

$$\int_{\mathbb{R}^d} \min\left(1, \|\xi\|^\alpha\right) \left|f(\xi)\right|^\alpha \, \mathrm{d}\xi < +\infty, \tag{1.5}$$

where $\|\cdot\|$ denotes the Euclidian norm on $\mathbb{R}^d$. Notice that, by analogy with the Gaussian case (see Bonami and Estrade (2003), for instance), the function $|f|^\alpha$ is called *the spectral density of the field* X.

Thanks to (1.5), for any $t \in \mathbb{R}^d$, the function $\xi \mapsto \left(e^{it\cdot\xi} - 1\right) f(\xi)$ belongs to $L^\alpha(\mathbb{R}^d)$, and thus it is integrable with respect to $\widetilde{M}_\alpha$. The field $\left\{X(t), t \in \mathbb{R}^d\right\}$ is defined, for all $t \in \mathbb{R}^d$, as

$$X(t) = X[f](t) := \mathcal{R}e\left\{\int_{\mathbb{R}^d} \left(e^{it\cdot\xi} - 1\right) f(\xi) \, \mathrm{d}\widetilde{M}_\alpha(\xi)\right\}, \tag{1.6}$$

where $t \cdot \xi$ denotes the usual inner product of t and ξ.

Harmonizable and Linear Fractional Stable Motions

A classical particular case: *harmonizable fractional stable motion (hfsm)* of Hurst parameter $H \in (0,1)$, denoted by $\{X^{\text{hfsm}}(t), t \in \mathbb{R}\}$ and defined as: $\forall\ t \in \mathbb{R}$,

$$X^{\text{hfsm}}(t) := \mathcal{R}e\left\{\int_{\mathbb{R}} \left(e^{it\xi} - 1\right)|\xi|^{-H-1/\alpha}\, \mathrm{d}\widetilde{M}_\alpha(\xi)\right\}. \tag{1.7}$$

X^{hfsm} is one of the two most classical extensions of the well-known *fractional Brownian motion (fBm)* to the setting of the heavy-tailed stable distributions.

The other classical extension of fBm to this setting is called *linear fractional stable motion (lfsm)* and denoted by $\left\{Y^{\text{lfsm}}(t), t \in \mathbb{R}^d\right\}$. In contrast with X^{hfsm}, the process Y^{lfsm} is not defined through an integral in the frequency domain but through an integral in the time domain:

$$Y^{\text{lfsm}}(t) := \int_{\mathbb{R}} \left(|t+s|^{H-1/\alpha} - |s|^{H-1/\alpha}\right) \mathrm{d}M_\alpha(s)\,, \tag{1.8}$$

where M_α is an α-stable real-valued random measure on $\mathbb{R}$.

In spite of the fact that the stable processes X^{hfsm} and Y^{lfsm} extend the same Gaussian process (the fBm) there are hudge differences between these two stable processes. For instance:

→ Sample paths of Y^{lfsm} are continuous functions only when $H > 1/\alpha$; in the latter case their critical Hölder regularity is $H - 1/\alpha$. This result can be derived from Kolmogorov's Hölder continuity Theorem, since, for each fixed $\gamma \in (0, \alpha)$, one has:

$$\forall\, t_1, t_2 \in \mathbb{R}, \quad \mathbb{E}\Big(\big|Y^{\text{lfsm}}(t_1) - Y^{\text{lfsm}}(t_2)\big|^{\gamma}\Big) = c_{\alpha,\gamma}\big|t_1 - t_2\big|^{1+\gamma(H-1/\gamma)}. \qquad (1.9)$$

→ Sample paths of X^{hfsm} are always continuous functions and their Hölder regularity is $H - \eta$, for any $\eta > 0$. This result can not be derived from Kolmogorov's Hölder continuity Theorem, even if X^{hfsm} also satisfies (1.9). It was obtained by Kôno and Maejima (1991) thanks to a LePage type series representation of X^{hfsm}.

Generally speaking there are hudge differences between stable stochastic fields defined through stochastic integrals in the frequency domain, and those defined through stochastic integrals in the time domain.

Our two motivations

In the continuous case, wavelet methods (see Ayache, Roueff and Xiao (2009)) have turned out to be efficient in the fine study of global and directional sample path behaviour of linear fractional stable sheet; that is the extension to $\mathbb{R}^d$ of the process Y^{lfsm}. Can this methodology be adapted to the general stationary increments stable field X? This issue is the main motivation of our talk.

Also we mention that the study of global and directional sample path behaviour of X may have an impact on future development of new applications related with modelling of anisotropic materials in frames of heavy-tailed stable distributions. It is worthwhile to note that in Gaussian frames such a modelling has already proved to be useful, in particular for detecting osteoporosis in human bones through the analysis of their radiographic images (see for instance Bonami and Estrade (2003) or Biermé, Richard, Rachidi and Benhamou (2009)).

Typically, X is an anisotropic model when the rate of vanishing at infinity of the corresponding spectral density $|f|^{\alpha}$ changes from one axis of $\mathbb{R}^d$ to another; therefore, we focus on the class $\mathcal{A}$ of the so-called admissible functions f, defined in the following way.

We set $p_* := \max\left\{2, \lfloor 1/\alpha \rfloor + 1\right\}$.
A function f belongs to $\mathcal{A}$ when it satisfies $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$.

$(\mathcal{H}_1)$ For all multi-index $p := (p_1, p_2, \ldots, p_d) \in \left\{0, 1, 2, \ldots, p_*\right\}^d$, the partial derivative function

$$\partial^p f := \frac{\partial^{p_1} \partial^{p_2} \ldots \partial^{p_d}}{(\partial \xi_1)^{p_1} (\partial \xi_2)^{p_2} \ldots (\partial \xi_d)^{p_d}} f \quad \text{(with the convention that } \partial^0 f := f\text{)}$$

is well-defined and continuous on the open set $\left(\mathbb{R} \setminus \{0\}\right)^d$; that is the Cartesian product of $\mathbb{R} \setminus \{0\}$ with itself d times.

$(\mathcal{H}_2)$ There are a positive constant c' and an exponent $a' \in (0,1)$ such that, for each $p \in \left\{0, 1, 2, \ldots, p_*\right\}^d$, and $\xi \in \left(\mathbb{R} \setminus \{0\}\right)^d$,

$$\|\xi\| \leq 1 \Longrightarrow \left|\partial^p f(\xi)\right| \leq c' \, \|\xi\|^{-a' - d/\alpha - \mathrm{l}(p)}, \tag{1.10}$$

where $\mathrm{l}(p) := p_1 + p_2 + \cdots + p_d$ is the length of the multi-index p.

$(\mathcal{H}_3)$ There exist a positive constant c and d positive exponents $a_1, \ldots, a_d$ such that for every $p \in \left\{0, 1, 2, \ldots, p_*\right\}^d$, and $\xi \in \left(\mathbb{R} \setminus \{0\}\right)^d$,

$$\|\xi\| \geq 1 \Longrightarrow \left|\partial^p f(\xi)\right| \leq c \prod_{l=1}^{d} (1 + |\xi_l|)^{-a_l - 1/\alpha - p_l}. \tag{1.11}$$

Organization of the talk

The Gaussian case $\alpha = 2$

We denote by $\{\psi_{J,K} : (J, K) \in \mathbb{Z}^d \times \mathbb{Z}^d\}$ the orthonormal basis of $L^2(\mathbb{R}^d)$ defined in the following way: for all $(J, K) := (j_1, \ldots, j_d, k_1, \ldots, k_d) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $x := (x_1, \ldots, x_d) \in \mathbb{R}^d$

$$\psi_{J,K}(x) := \prod_{l=1}^{d} 2^{j_l/2}\psi^1(2^{j_l}x_l - k_l), \tag{2.1}$$

where ψ^1 denotes an usual 1D Lemarié-Meyer mother wavelet. We refer to the book of Meyer (1990) and to that of Daubechies (1992) for a complete description of the wavelet tools used in the present section. It is worthwhile noting that ψ^1 is a real-valued function belonging to the Schwartz class $S(\mathbb{R})$, and that its Fourier transform $\widehat{\psi^1}$ is a compactly supported C^∞ function on $\mathbb{R}$, such that

$$\operatorname{supp}\widehat{\psi^1} \subseteq \left\{\lambda \in \mathbb{R} : \frac{2\pi}{3} \le |\lambda| \le \frac{8\pi}{3}\right\}. \tag{2.2}$$

The fact that the Fourier transform map is an isometry from $L^2(\mathbb{R}^d)$ into itself implies that "the Fourier transform of the basis $\{\psi_{J,K} : (J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d\}$", that is $\{\widehat{\psi}_{J,K} : (J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d\}$, is also an orthonormal basis of $L^2(\mathbb{R}^d)$. Thus, for any fixed $t \in \mathbb{R}^d$, the kernel function $\xi \mapsto \left(e^{it\cdot\xi} - 1\right) f(\xi)$, associated with $X(t)$, can be expressed as:

$$\left(e^{it\cdot\xi} - 1\right) f(\xi) = \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} s_{J,K}(t)\overline{\widehat{\psi}_{J,K}(\xi)} \qquad (\text{in } L^2(\mathbb{R}^d)). \tag{2.3}$$

The coefficients $s_{J,K}(t)$ are given by

$$s_{J,K}(t) := \int_{\mathbb{R}^d} \left(e^{it\cdot\xi} - 1\right) f(\xi) \widehat{\psi}_{J,K}(\xi)\,\mathrm{d}\xi = \Psi_J\left(2^J t - K\right) - \Psi_J\left(-K\right), \tag{2.4}$$

where, for all $x \in \mathbb{R}^d$,

$$\Psi_J(x) := 2^{(j_1+\cdots+j_d)/2} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f\left(2^J\xi\right) \widehat{\psi}_{0,0}(\xi)\mathrm{d}\xi, \tag{2.5}$$

with the convention that $2^J\xi := (2^{j_1}\xi_1, \ldots, 2^{j_d}\xi_d)$.

Therefore, we get that

$$X(t)=\mathcal{R}e\left\{\int_{\mathbb{R}^d}\Big(\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\big(\Psi_J\left(2^J t-K\right)-\Psi_J\left(-K\right)\big)\overline{\widehat{\psi}_{J,K}(\xi)}\Big)\,\mathrm{d}\widetilde{M}_2(\xi)\right\}. \tag{2.6}$$

Finally, in view of the isometry property of the stochastic integral $\int_{\mathbb{R}^d}\left(\,\cdot\,\right)\mathrm{d}\widetilde{M}_2$, it turns out that one can interchange in (2.6) the integration and the summation. Thus, we obtain that

$$X(t)=\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d}\big(\Psi_J\left(2^J t-K\right)-\Psi_J\left(-K\right)\big)\epsilon_{J,K}, \tag{2.7}$$

where the series converges in $L^2(\Omega)$, and the $\epsilon_{J,K}$'s are the independent $\mathcal{N}(0,1)$ Gaussian random variables

$$\epsilon_{J,K}:=\mathcal{R}e\left\{\int_{\mathbb{R}^d}\overline{\widehat{\psi}_{J,K}(\xi)}\,\mathrm{d}\widetilde{M}_2(\xi)\right\}. \tag{2.8}$$

The general case $\alpha \in (0,2)$

The arguments, we have used in the "convenient" framework of the Hilbert space $L^2(\mathbb{R}^d)$, have to be adapted to the "more hostile" framework of the space $L^\alpha\big(\mathbb{R}^d\big)$.

The main difficulty comes from the fact that $\big\{\widehat{\psi}_{J,K} : (J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d\big\}$ is no longer a basis of $L^\alpha\big(\mathbb{R}^d\big)$.

The function $\psi_{\alpha,J,K} = 2^{(j_1+\cdots+j_d)(1/2-1/\alpha)}\,\psi_{J,K}$ denotes the renormalized version of the function $\psi_{J,K}$ so that $\|\widehat{\psi}_{\alpha,J,K}\|_{L^\alpha(\mathbb{R}^d)}$ does not depend on (J,K).

Thus setting $\Psi_{\alpha,J} = 2^{(j_1+\cdots+j_d)(1/\alpha-1/2)}\Psi_J$, it follows that for every $(J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $(t,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, one has

$$\begin{aligned} s_{J,K}(t)\overline{\widehat{\psi}_{J,K}(\xi)} &= \big(\Psi_J\left(2^J t - K\right) - \Psi_J\left(-K\right)\big)\overline{\widehat{\psi}_{J,K}(\xi)} \\ &= \big(\Psi_{\alpha,J}\left(2^J t - K\right) - \Psi_{\alpha,J}\left(-K\right)\big)\overline{\widehat{\psi}_{\alpha,J,K}(\xi)}. \end{aligned} \tag{2.9}$$

Recall that $L^\alpha\big(\mathbb{R}^d\big)$ is a complete metric space for the distance

$$D_\alpha(g_1,g_2) := \begin{cases} \|g_1-g_2\|_{L^\alpha(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |g_1(\xi)-g_2(\xi)|^\alpha \,\mathrm{d}\xi\Big)^{1/\alpha}, & \text{if } \alpha \geq 1, \\ \|g_1-g_2\|^\alpha_{L^\alpha(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |g_1(\xi)-g_2(\xi)|^\alpha \,\mathrm{d}\xi, & \text{else.} \end{cases} \tag{2.10}$$

Also, notice $D_\alpha(g_1,g_2) = D_\alpha(g_1-g_2,0)$.

The proof of the fact that, for any fixed $t \in \mathbb{R}^d$, the kernel function $\xi \mapsto \big(e^{it\cdot\xi}-1\big) f(\xi)$, associated with $X(t)$, can be expressed as:

$$\big(e^{it\cdot\xi}-1\big) f(\xi) = \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} s_{J,K}(t)\overline{\widehat{\psi}_{J,K}(\xi)} \qquad (\text{in } L^\alpha(\mathbb{R}^d)), \tag{2.11}$$

is divided in two steps.

Step 1. We show that

$$\sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} D_\alpha\left(s_{J,K}(t)\overline{\widehat{\psi}_{\alpha,J,K}(\cdot)},0\right) < +\infty. \qquad (2.12)$$

Notice that, in view of the completeness of $L^\alpha(\mathbb{R}^d)$, (2.12) implies that there exists $F(t,\cdot)$ in $L^\alpha(\mathbb{R}^d)$ such that

$$F(t,\xi) = \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} s_{J,K}(t)\overline{\widehat{\psi}_{J,K}(\xi)} \qquad (\text{in } L^\alpha(\mathbb{R}^d)). \qquad (2.13)$$

Step 2. We show that, for all $t\in\mathbb{R}^d$ and almost all $\xi\in\mathbb{R}^d$,

$$F(t,\xi) = \left(e^{it\cdot\xi}-1\right)f(\xi). \qquad (2.14)$$

Basically the Step 2 is derived from the fact that, for any fixed arbitrarily small $\eta>0$, the function $\xi\mapsto\left(e^{it\cdot\xi}-1\right)f(\xi)\prod_{l=1}^d \mathbf{1}_{\{|\xi_l|\geq\eta\}}$ belongs to $L^2(\mathbb{R}^d)$.

Basically, the Step 1 is derived from the following proposition.

Proposition 2.1

$\Psi_{\alpha,J}$ is infinitely differentiable on $\mathbb{R}^d$. Moreover, all the functions $\partial^b \Psi_{\alpha,J}$, $b \in \mathbb{Z}_+^d$, are well-localized:

(i) *There is a positive constant c, such that for all $J \in \mathbb{Z}_+^d$, and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,*

$$\left|\partial^b \Psi_{\alpha,-J}(x)\right| \leq c\,\frac{\left(2^{-j_1} + \cdots + 2^{-j_d}\right)^{-a'-d/\alpha} \prod_{l=1}^d 2^{-j_l/\alpha}}{\prod_{l=1}^d (1+|x_l|)^{p_*}}, \qquad (2.15)$$

(ii) *For each $\zeta = (\zeta_1, \dots, \zeta_d) \in \{0,1\}^d \setminus \{(0,\dots,0)\}$, there exists a positive constant c, such that for every $J \in \prod_{l=1}^d \mathbb{Z}_{\zeta_l}$ ($\mathbb{Z}_1 = \mathbb{N}$ and $\mathbb{Z}_0 = \mathbb{Z}_-$) and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$,*

$$\left|\partial^b \Psi_{\alpha,J}(x)\right| \leq c \prod_{l=1}^d \frac{2^{(1-\zeta_l)j_l/\alpha}\, 2^{-j_l \zeta_l a_l}}{(1+|x_l|)^{p_*}}. \qquad (2.16)$$

Recall that $p_* := \max\left\{2, \lfloor 1/\alpha \rfloor + 1\right\}$.

Next, using arguments rather similar to those in the Gaussian case, we can show that:

Proposition 2.2 (Wavelet representation of X)

The field $\{X(t), t \in \mathbb{R}^d\}$ can be expressed, for each fixed $t \in \mathbb{R}^d$, as

$$X(t) = \sum_{(J,K)\in\mathbb{Z}^d\times\mathbb{Z}^d} \Big(\Psi_{\alpha,J}\left(2^J t - K\right) - \Psi_{\alpha,J}\left(-K\right)\Big)\epsilon_{\alpha,J,K}, \tag{2.17}$$

where the series converges in probability, and the $\epsilon_{\alpha,J,K}$'s are the identically distributed symmetric α-stable random variables

$$\epsilon_{\alpha,J,K} := \mathcal{R}e\left\{\int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)}\,\mathrm{d}\widetilde{M}_\alpha(\xi)\right\}. \tag{2.18}$$

→ The convergence of the series in (2.17) can be strenghtened to almost sure uniform convergence in t belonging to any compact subset of $\mathbb{R}^d$.
→ In contrast with the Gaussian case, the $\epsilon_{\alpha,J,K}$'s are not independent, they even have a complicated dependence structure.

Organization of the talk

Basically, path behaviour of X is determined by asymptotic behaviour of the sequence $\{\epsilon_{\alpha,J,K} : (J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d\}$. Let us state three crucial lemmas on this latter one.

Lemma 3.1 (the case $\alpha \in (0,1)$)

There exists an event Ω^* of probability 1 which depends on α and satisfies the following property: for all fixed $\delta \in (0,+\infty)$ and $\omega \in \Omega^*$, there is a finite constant $C(\omega) > 0$ (depending on α, δ and ω), such that, for every $J = (j_1, \dots, j_d) \in \mathbb{Z}^d$ and $K \in \mathbb{Z}^d$, one has

$$|\epsilon_{\alpha,J,K}(\omega)| \leq C(\omega) \prod_{l=1}^{d} (1 + |j_l|)^{1/\alpha+\delta}. \tag{3.1}$$

A rather surprising fact is that $|\epsilon_{\alpha,J,K}(\omega)|$ can be bounded independently on K when $\alpha \in (0,1)$.

The previous lemma and the next one are obtained by using a LePage series representation of the complex-valued α-stable process

$$\left\{ \int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)}\, \mathrm{d}\widetilde{M}_\alpha(\xi) : (J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d \right\}.$$

Lemma 3.2 (the case $\alpha \in [1,2)$)

There exists an event Ω^* of probability 1 which depends on α and satisfies the following property: for each fixed $\delta \in (0,+\infty)$ and $\omega \in \Omega^*$, there is a finite constant $C(\omega) > 0$ (depending on α, δ and ω), such that for all $(J,K) = (j_1,\ldots,j_d,k_1,\ldots,k_d) \in \mathbb{Z}^d \times \mathbb{Z}^d$,

$$|\epsilon_{\alpha,J,K}(\omega)| \leq C(\omega) \sqrt{\log\left(3 + \sum_{l=1}^{d} \left(|j_l| + |k_l|\right)\right)} \prod_{l=1}^{d} (1 + |j_l|)^{1/\alpha+\delta}. \qquad (3.2)$$

In contrast with the previous two lemmas, the following one is a rather classical result.

Lemma 3.3 (the Gaussian case $\alpha = 2$)

There exists an event Ω^* of probability 1 satisfying the following property: for every fixed $\omega \in \Omega^*$, there is a finite constant $C(\omega) > 0$ (depending on ω), such that for each $(J, K) = (j_1, \ldots, j_d, k_1, \ldots, k_d) \in \mathbb{Z}^d \times \mathbb{Z}^d$,

$$|\epsilon_{J,K}(\omega)| := |\epsilon_{2,J,K}(\omega)| \le C(\omega) \sqrt{\log\Big(3 + \sum_{l=1}^{d} (|j_l| + |k_l|)\Big)}. \tag{3.3}$$

Notice that in the three crucial lemmas, we have just stated, the events of full probability Ω^* are universal in the sense that they do not depend on the particular choice of the function f associated with the field X. The results, we will obtain, on path behaviour of X are valid on these universal events.

Sketch of the proof of Lemma 3.1: Let $\{\kappa^m : m \in \mathbb{N}\}$, $\{\Gamma_m : m \in \mathbb{N}\}$, and $\{g_m : m \in \mathbb{N}\}$ be three arbitrary mutually independent sequences of random variables having the following three properties.

1. The κ^m's, $m \in \mathbb{N}$, are $\mathbb{R}^d$-valued, independent, identically distributed and absolutely continuous, with a probability density function, denoted by ϕ, such that the measure $\phi(\xi)\mathrm{d}\xi$ is equivalent to the Lebesgue measure $\mathrm{d}\xi$ on $\mathbb{R}^d$.
2. The Γ_m's, $m \in \mathbb{N}$, are Poisson arrival times with unit rate.
3. The g_m's, $m \in \mathbb{N}$, are complex-valued, independent, identically distributed, rotationally invariant and satisfy $\mathbb{E}[|\mathcal{R}e(g_m)|^\alpha] = 1$.

LePage representation: there is a deterministic constant $a(\alpha) > 0$ such that

$$\left\{ \int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)}\, \mathrm{d}\widetilde{M}_\alpha(\xi) : (J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d \right\}$$

has the same distribution as

$$\left\{ a(\alpha) \sum_{m=1}^{+\infty} g_m \Gamma_m^{-1/\alpha} \phi(\kappa^m)^{-1/\alpha}\, \overline{\widehat{\psi}_{\alpha,J,K}(\kappa^m)} :\ (J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d \right\}.$$

From now on, these two processes are identified, also, we assume that the g_m's, $m \in \mathbb{N}$, are complex-valued centred Gaussian random variables, and that the probability density function ϕ is such that, for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d \setminus \{0\}$,

$$\phi(\xi) := \left(\frac{\eta}{4}\right)^d \prod_{l=1}^{d} |\xi_l|^{-1} \left(1 + |\log|\xi_l||\right)^{-1-\eta}, \tag{3.4}$$

where $\eta > 0$ is arbitrary fixed.

$\rightarrow$ Using the fact that, for all $(J, K) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and $\xi \in \mathbb{R}^d$,

$$\widehat{\psi}_{\alpha,J,K}(\xi) = \prod_{l=1}^{d} 2^{-j_l/\alpha} e^{-i2^{-j_l}k_l\xi_l}\, \widehat{\psi^1}(2^{-j_l}\xi_l), \tag{3.5}$$

we get, for some deterministic constant c_1, not depending on (J, K) and m, that

$$\begin{aligned} &\phi(\kappa^m)^{-1/\alpha} \left|\overline{\widehat{\psi}_{\alpha,J,K}(\kappa^m)}\right| \\ &\leq \left(\frac{\eta}{4}\right)^{-d/\alpha} \prod_{l=1}^{d} \left|2^{-j_l}\kappa_l^m\right|^{1/\alpha} \left(1 + |j_l| + \left|\log\left|2^{-j_l}\kappa_l^m\right|\right|\right)^{(1+\eta)/\alpha} \left|\widehat{\psi^1}(2^{-j_l}\kappa_l^m)\right| \\ &\leq c_1 \prod_{l=1}^{d} (1 + |j_l|)^{(1+\eta)/\alpha}. \end{aligned} \tag{3.6}$$

$\rightarrow$ In view of the Gaussianity assumption on the g_m's, $m \in \mathbb{N}$, it can be derived from the Borel-Cantelli's Lemma that, almost surely, for all $m \in \mathbb{N}$, one has

$$|g_m| \leq C_2 \sqrt{\log(3+m)}, \tag{3.7}$$

where C_2 is a finite random variable not depending on (J, K) and m.

$\rightarrow$ It results from the strong law of large number, that almost surely, for any $m \in \mathbb{N}$, the Poisson arrival time Γ_m satisfies

$$C_3 m \leq \Gamma_m \leq C_4 m, \tag{3.8}$$

where C_3 and C_4 are two positive finite random variables not depending on (J, K) and m.

Finally, it follows from (3.6) to (3.8) that, almost surely, for all $(J,K) \in \mathbb{Z}^d \times \mathbb{Z}^d$, one has

$$\begin{aligned} \Big| \int_{\mathbb{R}^d} \overline{\widehat{\psi}_{\alpha,J,K}(\xi)}\, \mathrm{d}\widetilde{M}_\alpha(\xi) \Big| &\leq a(\alpha) \sum_{m=1}^{+\infty} |g_m|\, \Gamma_m^{-1/\alpha} \phi(\kappa^m)^{-1/\alpha} \Big| \overline{\widehat{\psi}_{\alpha,J,K}(\kappa^m)} \Big| \\ &\leq C_5 \prod_{l=1}^{d} (1+|j_l|)^{(1+\eta)/\alpha}, \end{aligned} \tag{3.9}$$

where the random variable

$$C_5 := a(\alpha) c_1\, C_2\, C_3^{-1/\alpha} \sum_{m=1}^{+\infty} m^{-1/\alpha} \sqrt{\log(3+m)} \tag{3.10}$$

is almost surely finite since $\alpha \in (0,1)$. $\square$

Let us now turn to the statements of our results on path behaviour of X. First, we mention that we will consider directional increments of X in a generalized sense, more precisely:

For every fixed $k \in \{1, \dots, d\}$, and $h_k \in \mathbb{R}$, we denote by $\Delta^k_{h_k}$, the operator from the space of the real-valued functions on $\mathbb{R}^d$, into itself; so that, when g is such a function, $\Delta^k_{h_k} g$ is then the function defined, for all $x \in \mathbb{R}^d$ as,

$$\big(\Delta^k_{h_k} g\big)(x) = g(x + h_k e_k) - g(x), \tag{3.11}$$

e_k being the vector of $\mathbb{R}^d$ whose k-th coordinate equals 1 and the others vanish.

Notice that the operators $\Delta^k_{h_k}$ are commutative, in the sense that, for all $(k, k') \in \{1, \dots, d\}^2$ and $(h_k, h'_{k'}) \in \mathbb{R}^2$, one has,

$$\Delta^{k'}_{h'_{k'}} \circ \Delta^k_{h_k} = \Delta^k_{h_k} \circ \Delta^{k'}_{h'_{k'}},$$

where the symbole "$\circ$" denotes the usual composition of operators. For every $h = (h_1, \dots, h_d) \in \mathbb{R}^d$ and multi-index $B = (b_1, \dots, b_d) \in \mathbb{Z}^d_+$, we denote by $\Delta^B_{(h)}$, the operator from the space of the real-valued functions on $\mathbb{R}^d$, into itself, defined by

$$\Delta^B_{(h)} := \Delta^{1,b_1}_{h_1} \circ \cdots \circ \Delta^{d,b_d}_{h_d}, \tag{3.12}$$

where, for all $k \in \{1, \dots, d\}$, $\Delta^{k,b_k}_{h_k}$ is $\Delta^k_{h_k}$ composed with itself b_k times, with the convention that $\Delta^{k,0}_{h_k}$ is the identity.

Definition 3.1 (it concerns powers of logarithmic factors in the next theorem)

(i) *We denote by* $\mathcal{L}_2$ *the function defined, for each* $(a,b)\in\mathbb{R}_+^2$*, as*

$$\mathcal{L}_2(a,b) := 1/2\,1_{\{b\geq a\}} + 1_{\{b=a\}}. \tag{3.13}$$

More precisely, one has: $\mathcal{L}_2(a,b)=0$ *if* $a>b$, $\mathcal{L}_2(a,b)=3/2$ *if* $a=b$*, and* $\mathcal{L}_2(a,b)=1/2$ *if* $a<b$.

(ii) *For any fixed* $\alpha\in(0,2)$*, we denote by* $\mathcal{L}_\alpha$ *the function defined, for each* $(a,b,\delta)\in\mathbb{R}_+^3$*, as*

$$\mathcal{L}_\alpha(a,b,\delta) := \big(1/\alpha + \lfloor\alpha\rfloor/2 + \delta\big)1_{\{b\geq a\}} + 1_{\{b=a\}}, \tag{3.14}$$

where $\lfloor\alpha\rfloor$ *is the integer part of* α*. More precisely,*

- *when* $\alpha\in(0,1)$*, one has:* $\mathcal{L}_\alpha(a,b,\delta)=0$ *if* $a>b$, $\mathcal{L}_\alpha(a,b,\delta)=1/\alpha+1+\delta$ *if* $a=b$*, and* $\mathcal{L}_\alpha(a,b,\delta)=1/\alpha+\delta$ *if* $a<b$;
- *when* $\alpha\in[1,2)$*, one has:* $\mathcal{L}_\alpha(a,b,\delta)=0$ *if* $a>b$, $\mathcal{L}_\alpha(a,b,\delta)=1/\alpha+3/2+\delta$ *if* $a=b$*, and* $\mathcal{L}_\alpha(a,b,\delta)=1/\alpha+1/2+\delta$ *if* $a<b$.

Theorem 3.1 (directional behaviour of X)

Let $a_1, \ldots, a_d$ be the exponents governing the asymptotic behaviour at infinity of f along the axes of $\mathbb{R}^d$. Moreover we assume that $B \in \mathbb{Z}_+^d$, $T \in (0, +\infty)$ and $\omega \in \Omega^*$ are arbitrary an fixed.

(i) *When $\alpha = 2$, one has*

$$\sup_{h \in [-T,T]^d} \left\{ \frac{\left\| \Delta_{(h)}^B X(\cdot, \omega) \right\|_{T,\infty}}{\prod_{l=1}^d |h_l|^{\min(b_l, a_l)} \left(\log \left(3 + |h_l|^{-1} \right) \right)^{\mathcal{L}_2(a_l, b_l)}} \right\} < +\infty. \tag{3.15}$$

(ii) *When $\alpha \in (0, 2)$, for all arbitrarily small positive real number δ, one has*

$$\sup_{h \in [-T,T]^d} \left\{ \frac{\left\| \Delta_{(h)}^B X(\cdot, \omega) \right\|_{T,\infty}}{\prod_{l=1}^d |h_l|^{\min(b_l, a_l)} \left(\log \left(3 + |h_l|^{-1} \right) \right)^{\mathcal{L}_\alpha(a_l, b_l, \delta)}} \right\} < +\infty. \tag{3.16}$$

Theorem 3.2 (behaviour of X at infinity)

Let $a' \in (0,1)$ be the exponent governing the behaviour of f in the vicinity of zero. Moreover we assume $\delta \in (0,+\infty)$ and $\omega \in \Omega^*$ are arbitrary and fixed.

1. *When $\alpha \in (0,1)$ one has*

$$\sup_{\|t\| \geq 1} \left\{ \|t\|^{-a'} \big(\log\left(3+\|t\|\right)\big)^{-d/\alpha-\delta} |X(t,\omega)| \right\} < +\infty. \tag{3.17}$$

2. *When $\alpha \in [1,2)$ one has*

$$\sup_{\|t\| \geq 1} \left\{ \|t\|^{-a'} \big(\log\left(3+\|t\|\right)\big)^{-d/\alpha-\delta} |X(t,\omega)| \right\} < +\infty. \tag{3.18}$$

3. *When $\alpha = 2$ one has*

$$\sup_{\|t\| \geq 1} \left\{ \|t\|^{-a'} \big(\log\log\left(3+\|t\|\right)\big)^{-1/2} |X(t,\omega)| \right\} < +\infty. \tag{3.19}$$