Entrance laws at the origin of self-similar Markov processes in $\mathbb{R}^d$

Batı Şengül
joint with Andreas Kyprianou, Victor Rivero and Loïc Chaumont

University of Bath

July 2016

A Markov process $X = (X_t : t \geq 0)$ on $\mathbb{R}^d$ is called *self-similar* with index $\alpha > 0$ if for every $c > 0$,

$$((cX_{c^{-\alpha}t} : t \geq 0), \mathbb{P}_x) \stackrel{d}{=} (X, \mathbb{P}_{cx}).$$

A Markov process $X = (X_t : t \geq 0)$ on $\mathbb{R}^d$ is called *self-similar* with index $\alpha > 0$ if for every $c > 0$,

$$((cX_{c^{-\alpha}t} : t \geq 0), \mathbb{P}_x) \stackrel{d}{=} (X, \mathbb{P}_{cx}).$$

Examples:

- Brownian motion is self-similar with $\alpha = 2$,
- Stable process are self-similar,
- Bessel processes are self-similar (but not Lévy!)

A Markov process $X = (X_t : t \geq 0)$ on $\mathbb{R}^d$ is called *self-similar* with index $\alpha > 0$ if for every $c > 0$,

$$((cX_{c^{-\alpha}t} : t \geq 0), \mathbb{P}_x) \stackrel{d}{=} (X, \mathbb{P}_{cx}).$$

Examples:

- Brownian motion is self-similar with $\alpha = 2$,
- Stable process are self-similar,
- Bessel processes are self-similar (but not Lévy!)

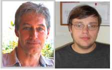
Set up:

- Self-similar process $((X, \mathbb{P}_x) : x \in \mathbb{R}^d \backslash \{0\})$,
- Killed after the first time $\tau_0 := \inf\{t \geq 0 : X_t = 0\}$ it hits the origin

Question

Does the limit $\lim_{x \to 0}(X, \mathbb{P}_x)$ *exist?*

Literature



BERTOIN & SAVOV (2010):
$\mathbb{R}_+$ using duality.

CHAUMONT, KYPRIANOU, PARDO, RIVERO (2012):
$\mathbb{R}_+$ using excursion theory

DEREICH, DÖRING, KYPRIANOU (2016):
$\mathbb{R}$ using Kuznetsov measures.

Skew decomposition and MAPs

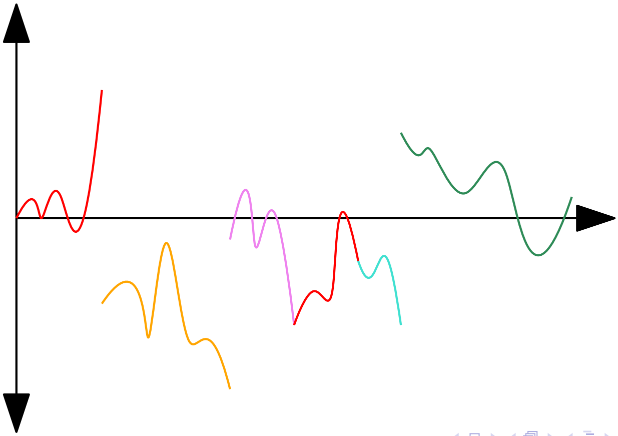
A Markov process $(((\xi_t, \Theta_t) : t \geq 0), \mathbf{P}_{r,\theta})$ on $\mathbb{R} \times \mathbb{S}^{d-1}$ is called a *Markov additive process*(MAP) if

$$\text{given } \Theta_t, \qquad (((\xi_{t+s} - \xi_t, \Theta_{t+s}) : s \geq 0), \mathbf{P}_{r,\theta}) \stackrel{d}{=} ((\xi, \Theta), \mathbf{P}_{0,\Theta_t}).$$

Skew decomposition and MAPs

A Markov process $(((\xi_t, \Theta_t) : t \geq 0), \mathbf{P}_{r,\theta})$ on $\mathbb{R} \times \mathbb{S}^{d-1}$ is called a *Markov additive process*(MAP) if

$$\text{given } \Theta_t, \qquad (((\xi_{t+s} - \xi_t, \Theta_{t+s}) : s \geq 0), \mathbf{P}_{r,\theta}) \stackrel{d}{=} ((\xi, \Theta), \mathbf{P}_{0,\Theta_t}).$$



Theorem (Alili, Chaumont, Graczyk, Zak (2016))

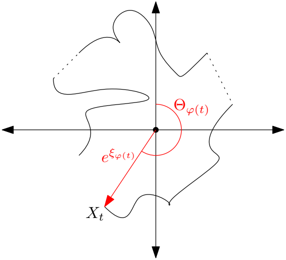
Suppose that $(X, \mathbb{P}_x)$ *is a ssMp killed when it hits the origin, then there exists a MAP* $((\xi, \Theta), \mathbf{P}_{r,\theta})$ *such that*

$$(X, \mathbb{P}_x) = ((\Theta_{\varphi(\|x\|^{-\alpha}t)} e^{\xi_{\varphi(\|x\|^{-\alpha}t)}} : t \geq 0), \mathbf{P}_{\log \|x\|, \arg(x)})$$

where

$$\varphi(t) := \inf \left\{ s > 0 : \int_0^s e^{\alpha \xi_u} \mathrm{d}u > t \right\}.$$

Conversely, for any MAP (ξ, Θ)*, the above transformation gives a ssMp.*



There are two separate problems

(i) Does there exist a process $(X, \mathbb{P}_0)$ started from the origin with the same transition rates as $(X, \mathbb{P})$?

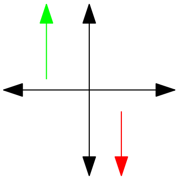
(ii) Is it true that $\lim_{x \to 0}(X, \mathbb{P}_x) = (X, \mathbb{P}_0)$?

There are two separate problems

(i) Does there exist a process $(X, \mathbb{P}_0)$ started from the origin with the same transition rates as $(X, \mathbb{P})$?

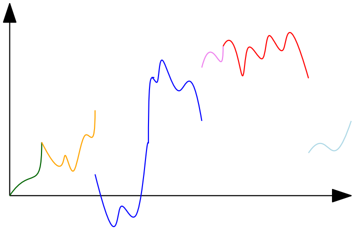
(ii) Is it true that $\lim_{x \to 0}(X, \mathbb{P}_x) = (X, \mathbb{P}_0)$?

Example: $\mathbb{R}^2$, move up unit speed in the positive half, move down unit speed negative half.

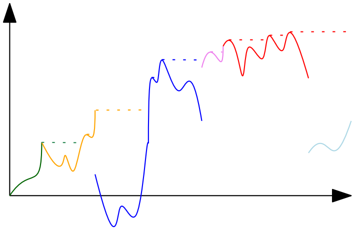


Has two entrance laws

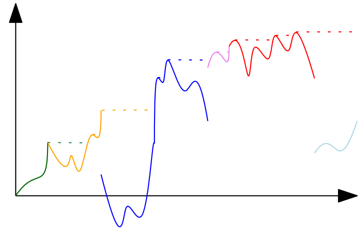
MAP ladder height (H_t^+, J_t^+):



MAP ladder height (H_t^+, J_t^+):



MAP ladder height (H^+_t, J^+_t):



Assumptions:

There exists a measure π such that

$$\mathbf{E}_{0,\pi}[H^+_1] < \infty. \qquad \text{(H)}$$

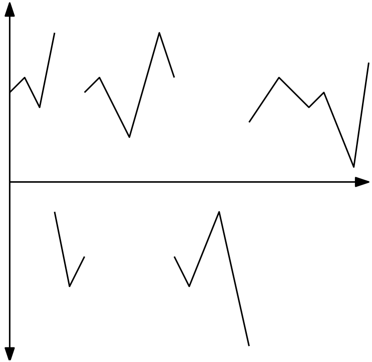
$$\xi \text{ is non-lattice.} \qquad \text{(NL)}$$

$$\mathbf{P}_{0,\theta}(\xi_t \in \mathrm{d}z; \Theta_t \in d\vartheta)\pi(\mathrm{d}\theta) = \mathbf{P}_{0,\vartheta}(\xi_t \in \mathrm{d}z; \Theta_t \in d\theta)\pi(\mathrm{d}\vartheta) \qquad \text{(WR)}$$

(π is necessarily invariant for Θ).

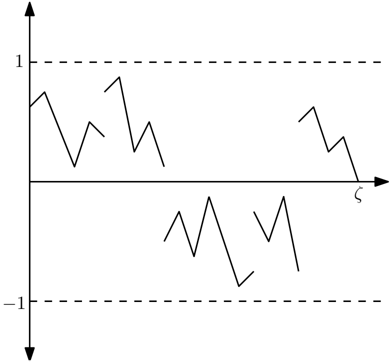
Theorem (Chaumont, Kyprianou, Rivero, Ş.)

Suppose that assumptions (H), (NL), (WR) hold, then there exists a process $(X, \mathbb{P}_0)$ *such that* $X_0 = 0$ *and* X *has the same transition rates as* $(X, \mathbb{P}_x)$.
Moreover explicit construction.



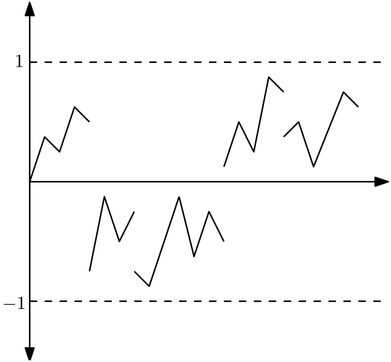
Theorem (Chaumont, Kyprianou, Rivero, Ş.)

Suppose that assumptions (H), (NL), (WR) hold, then there exists a process $(X, \mathbb{P}_0)$ *such that* $X_0 = 0$ *and* X *has the same transition rates as* $(X, \mathbb{P}_x)$.
Moreover explicit construction.



Theorem (Chaumont, Kyprianou, Rivero, Ş.)

Suppose that assumptions (H), (NL), (WR) hold, then there exists a process $(X, \mathbb{P}_0)$ *such that* $X_0 = 0$ *and* X *has the same transition rates as* $(X, \mathbb{P}_x)$*.*
Moreover explicit construction.



Define a new measure $\hat{\mathbf{P}}^{\downarrow}_{y,\theta}$ under which ξ is conditioned to remain below 0:

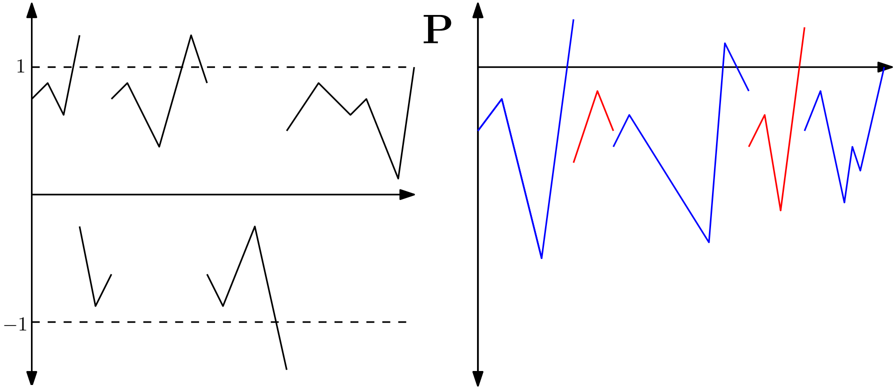
$$\frac{\mathrm{d}\hat{\mathbf{P}}^{\downarrow}_{y,\theta}}{\mathrm{d}\hat{\mathbf{P}}_{y,\theta}}\Big|_{\mathcal{F}_t} = \frac{\hat{U}^{+}_{\Theta_t}(\xi_t)}{\hat{U}^{+}_{\theta}(y)} 1_{\{T_0^+ > t\}}$$

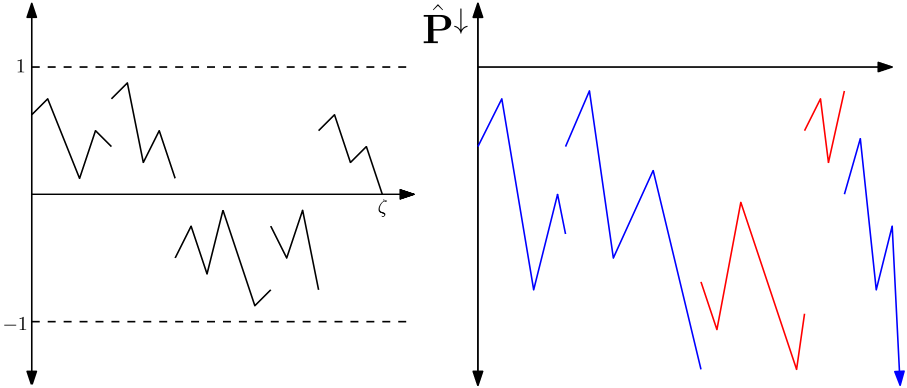
where $((\xi, \Theta), \hat{\mathbf{P}}) = ((-\xi, \Theta), \mathbf{P})$,

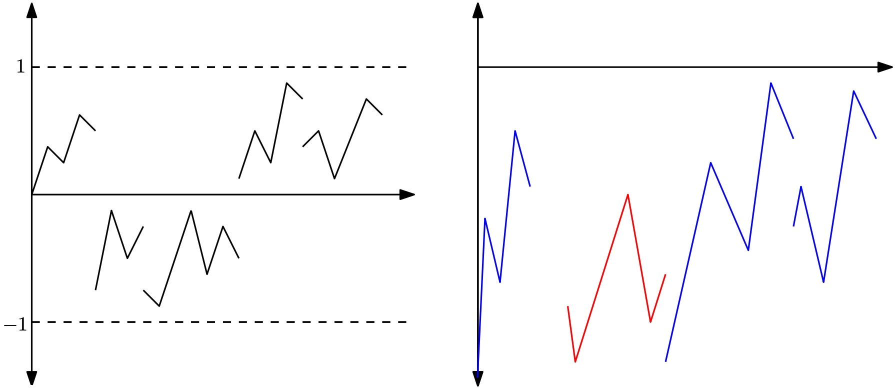
$$\hat{U}^{+}_{\theta}(y) = \hat{\mathbf{E}}_{0,\theta}\left[\int_0^\infty 1_{\{H_t^+ \le y\}}\,\mathrm{d}t\right]$$

and

$$T_0^+ := \inf\{t \ge 0 : \xi_t \ge 0\}.$$

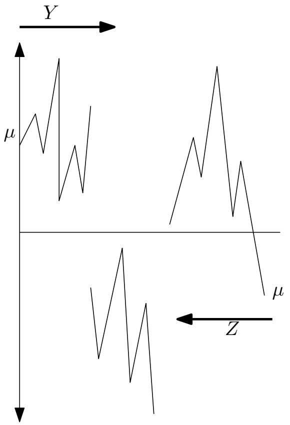

1
−1
P


1
−1
ζ
$\hat{\mathbf{P}}^{\downarrow}$


1
−1

Duality: $(Y, \mathbb{Q})$ is dual with $(Z, \mathbb{Q}')$ with respect to μ if

$$\int \mu(\mathrm{d}x) g(x) \mathbb{Q}_x[f(Y_t)] = \int \mu(\mathrm{d}x) f(x) \mathbb{Q}'_x[g(Z_t)]$$



We want to time-changed duality.

Time-changed MAP:

$(\xi^\varphi, \Theta^\varphi)$ started from (y, θ) by setting

$$(\xi^\varphi_t, \Theta^\varphi_t) = (\xi_{\varphi(\mathrm{e}^{-\alpha y} t)}, \Theta_{\varphi(\mathrm{e}^{-\alpha y} t)}) \qquad t < \bar{\zeta}$$

where

$$\bar{\zeta} = \bar{\zeta}(y) = \mathrm{e}^{\alpha y} \int_0^\infty \exp\{\xi_u\} \, \mathrm{d}u$$

is the total life time of the process.

Killed and time-changed MAP: $(\xi^{\dagger,\varphi}, \Theta^{\dagger,\varphi})$ the process of $(\xi^\varphi, \Theta^\varphi)$ killed after time $T_0^{\varphi,+} := \inf\{t \geq 0 : \xi^\varphi_t > 0\}$.

Using assumption (WR) show that.

Lemma

$((\xi^{\dagger,\varphi}, \Theta^{\dagger,\varphi}), \mathbf{P})$ and $((\xi^{\varphi}, \Theta^{\varphi}), \hat{\mathbf{P}}^{\downarrow})$ are in duality with respect to the measure

$$\nu(\mathrm{d}y, \mathrm{d}\theta) = \mathrm{e}^{-\alpha y} \hat{U}_{\theta}^{+}(y)\mathrm{d}y\pi(\mathrm{d}\theta)1_{\{y\leq 0\}}.$$

Nagasawa (1964), Chung and Walsh (2005)

Suppose that η satisfies

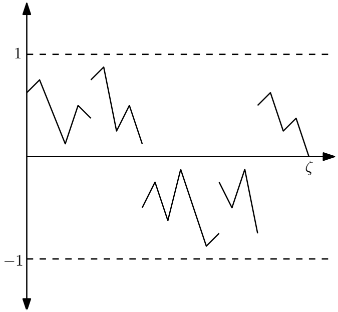
$$\int\int \nu(\mathrm{d}x, \mathrm{d}\theta) f(x,\theta) = \int\int \eta(\mathrm{d}x, \mathrm{d}\theta)\hat{\mathbf{E}}^{\downarrow}_{x,\theta}\left[\int_0^{\bar{\zeta}} f(\xi_t^{\varphi}, \Theta_t^{\varphi})\mathrm{d}t\right]$$

and $\bar{\zeta} < \infty$, $\hat{\mathbf{P}}^{\downarrow}$-almost surely.

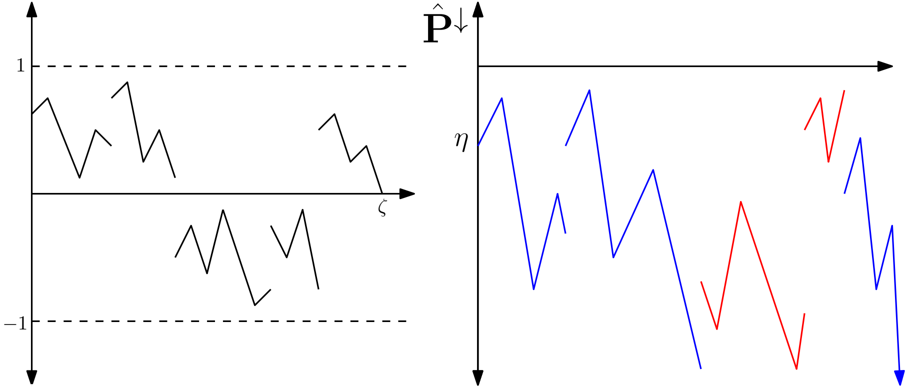
Then $(((\xi^{\varphi}_{(\bar{\zeta}-t)-}, \Theta^{\varphi}_{(\bar{\zeta}-t)-}) : t \leq \zeta), \hat{\mathbf{P}}^{\downarrow}_{\eta})$ has the same transition rates as $((\xi^{\dagger,\varphi}, \Theta^{\dagger,\varphi}), \mathbf{P})$.

Use the remaining assumptions (H) and (NL) to show that $\hat{\mathbf{P}}^{\downarrow}$ almost surely

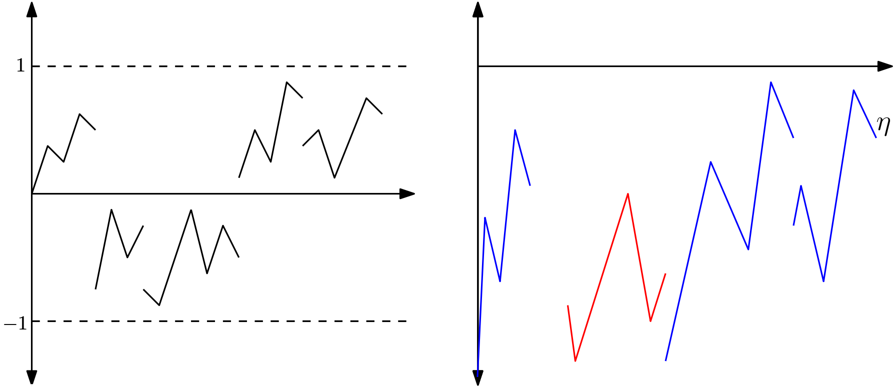
- $\bar{\zeta} < \infty$
- $\xi^{\varphi}_{\bar{\zeta}-} = -\infty$



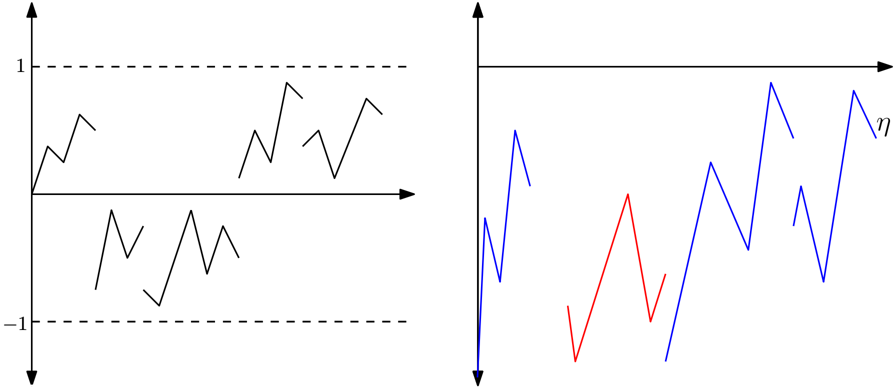
What is η?



What is η?



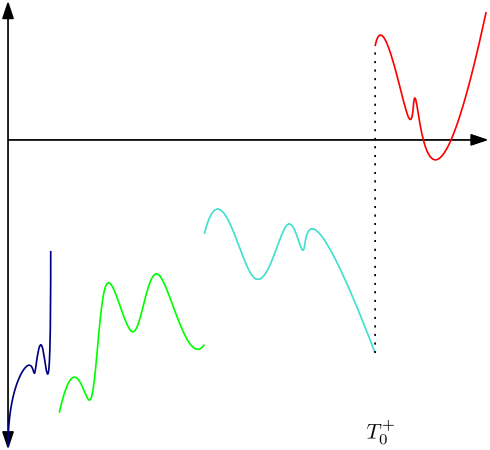
What is η?



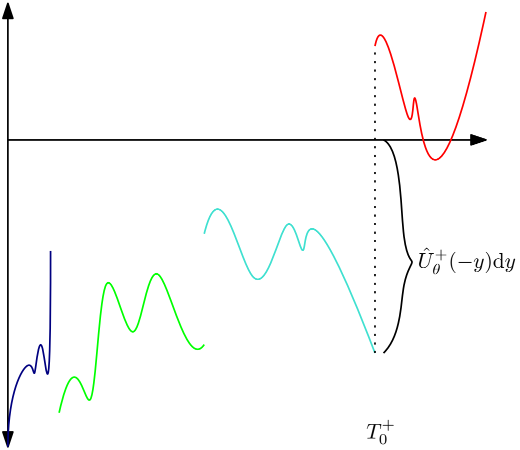
η should be asymptotic undershoot of the MAP:

$$\eta(\mathrm{d}x, \mathrm{d}\theta) = \lim_{y \downarrow -\infty} \mathbf{P}_{y,\pi}\left(\xi_{T_0^+ -} \in \mathrm{d}x; \Theta_{T_0^+ -} \in \mathrm{d}\theta\right)$$

$$\eta(\mathrm{d}y, \mathrm{d}\theta) = 1_{\{y \leq 0\}} \pi(\mathrm{d}\theta)$$



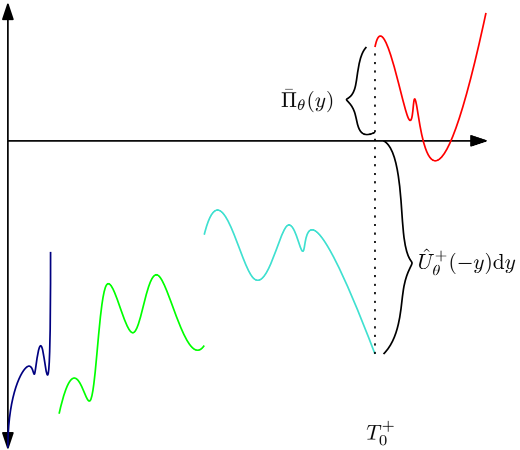
$$\eta(\mathrm{d}y, \mathrm{d}\theta) = 1_{\{y \leq 0\}} \pi(\mathrm{d}\theta) \mathrm{d}y \, \hat{U}_\theta^+(-y)$$



$$\eta(\mathrm{d}y, \mathrm{d}\theta) = 1_{\{y \leq 0\}} \pi(\mathrm{d}\theta) \mathrm{d}y \, \hat{U}^+_\theta(-y) \bar{\Pi}_\theta(y)$$

where

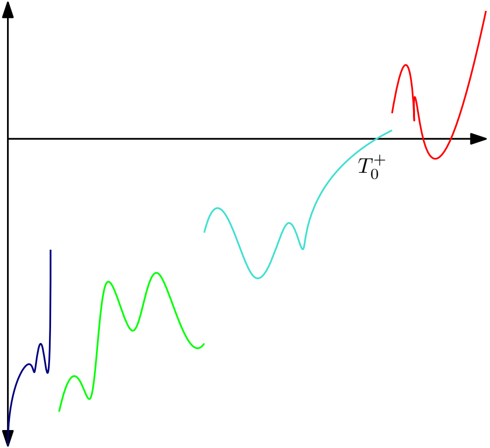
$$\bar{\Pi}_\theta(y) = \lim_{\delta \to 0} \mathbf{P}_{0,\delta}(\xi_\delta \leq y).$$



$$\eta(\mathrm{d}y, \mathrm{d}\theta) = 1_{\{y\leq 0\}}\pi(\mathrm{d}\theta)\mathrm{d}y\,\hat{U}^{+}_{\theta}(-y)\bar{\Pi}_{\theta}(y) + \mathrm{a}_{\theta}\pi(\mathrm{d}\theta)\delta_0(\mathrm{d}y)$$

where

$$\bar{\Pi}_{\theta}(y) = \lim_{\delta\to 0} \mathbf{P}_{0,\delta}(\xi_{\delta} \leq y).$$



Convergence to entrance law

A point $x \in \mathbb{R}^d \backslash \{0\}$ is called accessible, if for every $y \in \mathbb{R}^d \backslash \{0\}$ and every open neighbourhood $U \subset \mathbb{R}^d \backslash \{0\}$ containing x, there exists a $t \geq 0$ such that

$$\mathbb{P}_y(X_t \in U) > 0.$$

Assumption:

$$\{(X, \mathbb{P}_x) : x \in \mathbb{R}^d \backslash \{0\}\} \text{ is a Feller family with an accessible point.} \quad \text{(A)}$$

Convergence to entrance law

A point $x \in \mathbb{R}^d \backslash \{0\}$ is called accessible, if for every $y \in \mathbb{R}^d \backslash \{0\}$ and every open neighbourhood $U \subset \mathbb{R}^d \backslash \{0\}$ containing x, there exists a $t \geq 0$ such that

$$\mathbb{P}_y(X_t \in U) > 0.$$

Assumption:

$$\{(X, \mathbb{P}_x) : x \in \mathbb{R}^d \backslash \{0\}\} \text{ is a Feller family with an accessible point.} \qquad \text{(A)}$$

(A) $\implies$ unique invariant measure for Θ

(H)+(NL)+(WR)+(A) $\implies$ a unique entrance law $(X, \mathbb{P}_0)$.

Theorem (Chaumont, Kyprianou, Rivero, Ş.)

Suppose assumptions (H), (NL), (WR) and (A) hold, then in the sense of Skorokhod convergence,

$$\lim_{x\to 0}(X,\mathbb{P}_x) = (X,\mathbb{P}_0).$$

Thank you!