Distributional Representations & Dominance of a Lévy Process over its Maximal Jump Processes

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• joint work with
Yuguang Fan (School of Mathematics & Statistics, Uni Melbourne)
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• talk surveys
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intro

• we study relations between a Lévy process $X = (X_t)_{t \geq 0}$, its quadratic variation process $V = (V_t)_{t \geq 0}$, and its maximal jump processes, with particular interest in how these processes, and how positive and negative parts of the X process, interact.

• ratio of the process to its extremes in the random walk situation: Darling (1952), Arov & Bobrov (1960), Maller & Resnick (1984), Kesten & Maller ((1992), (1994)); almost sure versions of sum/max relationships, see Feller (1968), Kesten & Maller (1995), Pruitt (1987).

• trimmed sums concerning heavy tailed distributions: Csörgő, Haeusler & Mason (1988), Berkes & Horváth (2010), Berkes, Horváth & Schauer (2010), and Griffin & Pruitt (2013); Silvestrov & Teugels (2002) concerns sums and maxima of random walks and triangular arrays; Ladoucette & Teugels (2013) for an insurance application; connections to St. Petersburg game: Gut & Martin-Löf (2014) give a “maxtrimmed” version of the game, while Fukker, Györfi & Kevei (2015).

intro

• Relevant to our topic, includes that of Doney (2004), Andrew (2008), Bertoin (1997), Doney (2007)

• identities allow for possible ties in the order statistics of the jumps. point process versions are motivated by LePage (1980, 1981), LePage, Woodroofe & Zinn (1981), Mori (1984) for trimmed sums, Khinthine's inverse Lévy measure method (1937),... Rosiński (2001) summarises alternative series representations for Lévy processes.

notation

- let $X = (X_t)_{t\geq 0}$ be a real-valued Lévy process with canonical triplet (γ, σ^2, Π) and characteristic function $Ee^{\mathrm{i}\theta X_t} = e^{t\Psi(\theta)}$, $t \geq 0$, $\theta \in \mathbb{R}$, with characteristic exponent

$$\Psi(\theta) := \mathrm{i}\theta\gamma - \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}_*} (e^{\mathrm{i}\theta x} - 1 - \mathrm{i}\theta x \mathbf{1}_{\{|x|\leq 1\}})\Pi(\mathrm{d}x)$$

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• measures on $(0, \infty)$: $\Pi^{(+)}$ is Π restricted to $(0, \infty)$, $\Pi^{(-)}$ is $\Pi(-\cdot)$ restricted to $(0, \infty)$, and $\Pi^{|\cdot|} := \Pi^{(+)} + \Pi^{(-)}$
$\Delta\Pi(y) := \Pi\{\{y\}\}$, $y \in \mathbb{R}_*$, and $\Delta\overline{\Pi}(y) := \overline{\Pi}(y-) - \overline{\Pi}(y)$, $y > 0$

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- positive, negative and two-sided tails of Π are

$$\overline{\Pi}^{+}(x) := \Pi\{(x,\infty)\},\ \overline{\Pi}^{-}(x) := \Pi\{(-\infty,-x)\},\ \overline{\Pi}(x) := \overline{\Pi}^{+}(x)+\overline{\Pi}^{-}(x),$$

$\overline{\Pi}^{\leftarrow}(x) = \inf\{y > 0 : \overline{\Pi}(y) \leq x\}$, $x > 0$, denotes the right-continuous inverse of the nonincreasing function

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- one-sided and modulus trimmed versions of X are defined as

$$^{(r)}X_t := X_t - \sum_{i=1}^{r} \Delta X_t^{(i)} \text{ and } ^{(r)}\widetilde{X}_t := X_t - \sum_{i=1}^{r} \widetilde{\Delta X}_t^{(i)},$$

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• introduce two families of processes, indexed by $v > 0$, truncating jumps from sample paths of X_t and V_t, respectively. let $v, t > 0$. if $\overline{\Pi}(0+) = \infty$, then set

$$ \widetilde{X}_t^{v} := X_t - \sum_{0<s\leq t} \Delta X_s \, \mathbf{1}_{\{|\Delta X_s| \geq \overline{\Pi}^{\leftarrow}(v)\}} $$

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representation theorem

assume independent X, $\mathfrak{S}_r$, Y^+, Y^-, and Y,
where $r \in \mathbb{N}$, $\mathfrak{S}_r \sim$Gamma$(r, 1)$, $Y^\pm = (Y_t^\pm)_{t \geq 0}$ and $Y = (Y_t)_{t \geq 0}$ are independent Poisson processes with $EY_1^\pm = EY_1 = 1$

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(i) if $\overline{\Pi}^+(0+) = \infty$, for $t, v > 0$, let $\kappa(v) := \overline{\Pi}^+(\overline{\Pi}^{+,\leftarrow}(v)-) - v$, and $G_t^v := \overline{\Pi}^{+,\leftarrow}(v) Y_{t\kappa(v)}$ then, for $t > 0$, we have

$$\left({}^{(r)}X_t, \,, \Delta X_t^{(r)}\right) \stackrel{\mathrm{D}}{=} \left(X_t^v + G_t^v, \overline{\Pi}^{+,\leftarrow}(v)\right)\Big|_{v=\mathfrak{S}_r/t}.$$

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(ii) if $\overline{\Pi}(0+) = \infty$, then ,for $t, v > 0$, let

$$\kappa^\pm(v) := \left(\overline{\Pi}\left(\overline{\Pi}^\leftarrow(v)-\right) - v\right) \frac{\Delta\Pi\left(\pm\overline{\Pi}^\leftarrow(v)\right)}{\Delta\overline{\Pi}\left(\overline{\Pi}^\leftarrow(v)\right)} \mathbf{1}_{\{\Delta\overline{\Pi}(\overline{\Pi}^\leftarrow(v))\neq 0\}}$$

$$\widetilde{G}_t^v := \overline{\Pi}^\leftarrow(v)(Y^+_{t\kappa^+(v)} - Y^-_{t\kappa^-(v)})$$

then, for each $t > 0$, we have

$$\left({}^{(r)}\widetilde{X}_t, |\widetilde{\Delta X}_t^{(r)}|\right) \stackrel{\mathrm{D}}{=} \left(\widetilde{X}_t^v + \widetilde{G}_t^v, \overline{\Pi}^\leftarrow(v)\right)\Big|_{v=\mathfrak{S}_r/t}$$

order statistics with ties

let $\mathbb{X}$ be a Poisson point process (PPP) on $[0, \infty) \times \mathbb{R}_*$ with intensity measure $\mathrm{d}s \otimes \Pi(\mathrm{d}x)$

$$\mathbb{X} = \sum_s \delta_{(s, \Delta X_s)}$$

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PPPs of positive and negative jumps and jumps in modulus with intensity measures $\mathrm{d}s \otimes \Pi^{\pm, |\cdot|}(\mathrm{d}x)$, respectively,

$$\begin{aligned} \mathbb{X}^{\pm} &= \sum_s \mathbf{1}_{(0,\infty)}(\pm \Delta X_s) \delta_{(s, \pm \Delta X_s)} \\ \mathbb{X}^{|\cdot|} &= \mathbb{X}^+ + \mathbb{X}^- = \sum_s \delta_{(s, |\Delta X_s|)} \end{aligned}$$

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restrict processes to the time interval $s\in[0,t]$, $t>0$,

$$\mathbb{X}_t(\cdot):=\mathbb{X}([0,t]\times\mathbb{R}_*\cap\cdot) \qquad \text{and} \qquad \mathbb{X}_t^{\pm,|\cdot|}(\cdot)=\mathbb{X}^{\pm,|\cdot|}([0,t]\times(0,\infty)\cap\cdot)$$

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assume $\overline{\Pi}(0+) = \Pi(0,\infty) = \infty$, $\overline{\Pi}(x) := \Pi(0,\infty) < \infty$ and $t > 0$.

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• let $\widetilde{T}^{(1)}(\mathbb{X}_t)$ be randomly chosen, independently of $(X_t)_{t\geq 0}$, according to the discrete uniform distribution,

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• define $\widetilde{\Delta X}_t^{(1)} = \widetilde{\Delta X}^{(1)}(\mathbb{X}_t) := \Delta X_{\widetilde{T}^{(1)}(\mathbb{X}_t)}$. and maximum modulus trimmed point process on $[0,t] \times \mathbb{R}_*$ by

$$^{(1)}\widetilde{\mathbb{X}}_t := \mathbb{X}_t - \delta_{(\widetilde{T}^{(1)}(\mathbb{X}_t), \widetilde{\Delta X}_t^{(1)})}$$

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• let $r = 2, 3 \ldots$. iteratively, define $\widetilde{T}^{(r)}(\mathbb{X}_t) := \widetilde{T}^{(1)}(^{(r-1)}\widetilde{\mathbb{X}}_t)$ and $\widetilde{\Delta X}_t^{(r)} := \Delta X_{\widetilde{T}^{(r)}(\mathbb{X}_t)}$, and the *r–fold trimmed extremal process of modulus jumps* is then defined by

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similarly,

• assuming $\overline{\Pi}^{+}(0+) = \infty$, introduce in $[0, t] \times (0, \infty)$,

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such that $\Delta X_t^{(1)} \geq \cdots \geq \Delta X_t^{(r)}$ are the r^{th} largest order statistics of positive jumps of X sampled on time interval $[0, t]$

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• subtract points corresponding to large jumps, introduce the *r–fold trimmed extremal process of positive jumps* by

$$^{(r)}\mathbb{X}_t^+ := \mathbb{X}_t^+ - \sum_{1 \leq i \leq r} \delta_{(T^{(i)}(\mathbb{X}_t^+),\, \Delta X_t^{(i)})}$$

alternative representation

- let $(\mathfrak{U}_i)$, $(\mathfrak{U}'_i)$ and $(\mathfrak{E}_i)$ be independent, where $(\mathfrak{U}_i)$ and $(\mathfrak{U}'_i)$ are iid sequences of uniform rvs in $(0, 1)$, $(\mathfrak{E}_i)$ is i.i.d. sequence of exponential rvs with parameter $E\mathfrak{E}_i = 1$.

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- $\mathfrak{S}_r = \sum_{i=1}^r \mathfrak{E}_i$ is standard Gamma random walk, $r \in \mathbb{N}$.
- for $t > 0$

$$\mathbb{V}_t := \sum_{i\geq 1} \delta_{(t\mathfrak{U}_i,\, \mathfrak{S}_i/t)} \quad \text{and} \quad \mathbb{V}'_t := \sum_{i\geq 1} \delta_{(t\mathfrak{U}_i,\, \mathfrak{U}'_i,\, \mathfrak{S}_i/t)}$$

$\mathbb{V}_t$ and $\mathbb{V}'_t$ are homogeneous PPPs on $[0,t] \times (0,\infty)$ and $[0,t] \times (0,1) \times (0,\infty)$ with intensity measures $\mathrm{d}s \otimes \mathrm{d}v$ and $\mathrm{d}s \otimes \mathrm{d}u' \otimes \mathrm{d}v$, respectively.

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- for $r \in \mathbb{N}_0 := \{0,1,2\ldots\}$, introduce r–fold trimmed counterparts:

$${}^{(r)}\mathbb{V}_t := \sum_{i > r} \delta_{(t\mathfrak{U}_i, \mathfrak{S}_i/t)} \quad \text{and} \quad {}^{(r)}\mathbb{V}'_t := \sum_{i > r} \delta_{(t\mathfrak{U}_i,\, \mathfrak{U}'_i, \mathfrak{S}_i/t)}$$

transformations for $\overline{\Pi}^+(0+) = \infty$

- if $\overline{\Pi}^+(0+) = \infty$ then transform

$$(I, \overline{\Pi}^{+,\leftarrow}) : [0,t] \times (0,\infty) \to [0,\infty) \times (0,\infty)$$

such that

$$\begin{aligned} \mathbb{V}_t^{(I,\overline{\Pi}^{+,\leftarrow})} &:= \sum_{i \geq 1} \delta_{(t\mathfrak{U}_i, \overline{\Pi}^{+,\leftarrow}(\mathfrak{S}_i/t))} \\ {}^{(r)}\mathbb{V}_t^{(I,\overline{\Pi}^{+,\leftarrow})} &:= \sum_{i > r} \delta_{(t\mathfrak{U}_i, \overline{\Pi}^{+,\leftarrow}(\mathfrak{S}_i/t))}. \end{aligned}$$

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- with Radon-Nikodym $\mathrm{d}\Pi^{\pm} = g^{\pm}\mathrm{d}\Pi^{|\cdot|}$ write

$g^{\pm} : (0,\infty) \to (0,\infty)$ with $g^{+} + g^{-} \equiv 1$ and $\mathrm{d}\Pi^{\pm} = g^{\pm}\mathrm{d}\Pi^{|\cdot|}$
returning signs by $m : [0,t] \times (0,1) \times (0,\infty) \to [0,t] \times \mathbb{R}_*$

$$m(s,u',x) := \begin{cases} (s,x)\,, & \text{if } u' < g^{+}(x) \\ (s,-x)\,, & \text{if } u' \geq g^{+}(x) \end{cases}$$

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- map: $\mathbb{V}'_t \xrightarrow{(I,I,\overline{\Pi}^{\leftarrow})} \mathbb{V}'^{(I,I,\overline{\Pi}^{\leftarrow})}_t := \sum_{i\geq 1} \delta_{(t\mathfrak{U}_i, \mathfrak{U}'_i, \overline{\Pi}^{\leftarrow}(\mathfrak{S}_i/t))}$

$$\xrightarrow{m} \mathbb{V}'^{m\circ(I,I,\overline{\Pi}^{\leftarrow})}_t := \sum_{i\geq 1} \delta_{m(t\mathfrak{U}_i, \mathfrak{U}'_i, \overline{\Pi}^{\leftarrow}(\mathfrak{S}_i/t))}$$

lemma

let $t > 0$ and $r \in \mathbb{N}$.
(i) if $\overline{\Pi}^{+}(0+) = \infty$, then

$$\mathbb{X}_t^{+} \stackrel{\mathrm{D}}{=} \mathbb{V}_t^{(I,\overline{\Pi}^{+,\leftarrow})}$$

$$\left(T^{(i)}(\mathbb{X}_t^{+}), \Delta X_t^{(i)}\right)_{i\geq 1} \stackrel{\mathrm{D}}{=} \left(t\mathfrak{U}_i, \overline{\Pi}^{+,\leftarrow}(\mathfrak{S}_i/t)\right)_{i\geq 1}$$

$$\left\{\left(T^{(i)}(\mathbb{X}_t^{+}), \Delta X_t^{(i)}\right)_{1\leq i\leq r}, {}^{(r)}\mathbb{X}_t^{+}\right\} \stackrel{\mathrm{D}}{=} \left\{\left(t\mathfrak{U}_i, \overline{\Pi}^{+,\leftarrow}(\mathfrak{S}_i/t)\right)_{1\leq i\leq r}, {}^{(r)}\mathbb{V}_t^{(I,\overline{\Pi}^{+,\leftarrow})}\right\}.$$

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\left(T^{(i)}(\mathbb{X}_t^{+}), \Delta X_t^{(i)}\right)_{i\geq 1} &\stackrel{\mathrm{D}}{=} \left(t\mathfrak{U}_i, \overline{\Pi}^{+,\leftarrow}(\mathfrak{S}_i/t)\right)_{i\geq 1} \\
\left\{\left(T^{(i)}(\mathbb{X}_t^{+}), \Delta X_t^{(i)}\right)_{1\leq i\leq r}, {}^{(r)}\mathbb{X}_t^{+}\right\} &\stackrel{\mathrm{D}}{=} \\
&\left\{\left(t\mathfrak{U}_i, \overline{\Pi}^{+,\leftarrow}(\mathfrak{S}_i/t)\right)_{1\leq i\leq r}, {}^{(r)}\mathbb{V}_t^{(I,\overline{\Pi}^{+,\leftarrow})}\right\}.
\end{aligned}$$

(ii) if $\overline{\Pi}(0+) = \infty$, then

$$\begin{aligned}
\mathbb{X}_t &\stackrel{\mathrm{D}}{=} \mathbb{V'}_t^{m\circ(I,I,\overline{\Pi}^{\leftarrow})} \\
\left(\widetilde{T}^{(i)}(\mathbb{X}_t), \widetilde{\Delta X}_t^{(i)}\right)_{i\geq 1} &\stackrel{\mathrm{D}}{=} \left(m\left(t\mathfrak{U}_i, \mathfrak{U}'_i, \overline{\Pi}^{\leftarrow}(\mathfrak{S}_i/t)\right)\right)_{i\geq 1} \\
\left\{\left(\widetilde{T}^{(i)}(\mathbb{X}_t), \widetilde{\Delta X}_t^{(i)}\right)_{1\leq i\leq r}, {}^{(r)}\widetilde{\mathbb{X}}_t\right\} &\stackrel{\mathrm{D}}{=} \\
&\left\{\left(m(t\mathfrak{U}_i, \mathfrak{U}'_i, \overline{\Pi}^{\leftarrow}(\mathfrak{S}_i/t))\right)_{1\leq i\leq r}, {}^{(r)}\mathbb{V'}_t^{m\circ(I,I,\overline{\Pi}^{\leftarrow})}\right\}
\end{aligned}$$

representation theorem (PPP)

assume $\mathbb{X}$, $(\mathfrak{U}_i)$, $(\mathfrak{U}'_i)$, $\mathfrak{S}_r$, $Y^{\pm} = (Y^{\pm}(t))_{t \geq 0}$, $Y = (Y(t))_{t \geq 0}$, are independent, with $Y^{\pm}$ and Y standard Poisson processes.

representation theorem (PPP)

assume $\mathbb{X}$, $(\mathfrak{U}_i)$, $(\mathfrak{U}'_i)$, $\mathfrak{S}_r$, $Y^{\pm} = (Y^{\pm}(t))_{t\geq 0}$, $Y = (Y(t))_{t\geq 0}$, are independent, with $Y^{\pm}$ and Y standard Poisson processes.

for $x > 0$ delete points in $\mathbb{X}_t^+$ and $\mathbb{X}_t$ not lying in $[0,t] \times (0,x)$ and $[0,t] \times (-x,x)_*$:

$\mathbb{X}_t^{+\cdot<x} := \mathbb{X}^+([0,t]\times(0,x)\cap\cdot\,)$ and $\mathbb{X}_t^{|\cdot|<x} := \mathbb{X}([0,t]\times(-x,x)_*\cap\cdot\,)$.

representation theorem (PPP)

assume $\mathbb{X}$, $(\mathfrak{U}_i)$, $(\mathfrak{U}_i')$, $\mathfrak{S}_r$, $Y^{\pm} = (Y^{\pm}(t))_{t\geq 0}$, $Y = (Y(t))_{t\geq 0}$, are independent, with $Y^{\pm}$ and Y standard Poisson processes.

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(i) if $\overline{\Pi}^+(0+) = \infty$, then for all $t > 0$, $r \in \mathbb{N}$,
$\kappa(v) = \overline{\Pi}^+(\overline{\Pi}^{+,\leftarrow}(v)-) - v$

$$\left(\Delta X_t^{(r)}, {}^{(r)}\mathbb{X}_t^+\right) \stackrel{\mathrm{D}}{=} \left(\overline{\Pi}^{+,\leftarrow}(v),\ \mathbb{X}_t^{+\cdot<\overline{\Pi}^{+,\leftarrow}(v)} + \sum_{i=1}^{Y(t\kappa(v))} \delta_{(t\mathfrak{U}_i,\ \overline{\Pi}^{+,\leftarrow}(v))}\right)_{v=\mathfrak{S}_r/t}$$

representation theorem (PPP)

assume $\mathbb{X}$, $(\mathfrak{U}_i)$, $(\mathfrak{U}'_i)$, $\mathfrak{S}_r$, $Y^{\pm} = (Y^{\pm}(t))_{t\geq 0}$, $Y = (Y(t))_{t\geq 0}$, are independent, with $Y^{\pm}$ and Y standard Poisson processes.

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(ii) if $\overline{\Pi}(0+) = \infty$ then, for all $t > 0$, $r \in \mathbb{N}$,

$$\left(|\widetilde{\Delta X}_t^{(r)}|, {}^{(r)}\widetilde{\mathbb{X}}_t\right)$$

$$\stackrel{\mathrm{D}}{=} \left(\overline{\Pi}^{\leftarrow}(v), \mathbb{X}_t^{|\cdot|<\overline{\Pi}^{\leftarrow}(v)} + \sum_{i=1}^{Y^+(t\kappa^+(v))} \delta_{(t\mathfrak{U}_i, \overline{\Pi}^{\leftarrow}(v))} + \sum_{i=1}^{Y^-(t\kappa^-(v))} \delta_{(t\mathfrak{U}'_i, -\overline{\Pi}^{\leftarrow}(v))}\right)_{v=\mathfrak{S}_r/t}$$

representation theorem (LP revisited)

assume independent X, $\mathfrak{S}_r$, Y^+, Y^-, and Y,
where $r \in \mathbb{N}$, $\mathfrak{S}_r \sim$Gamma$(r,1)$, $Y^{\pm} = (Y^{\pm}_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ are independent Poisson processes with $EY_1^{\pm} = EY_1 = 1$.

(i) if $\overline{\Pi}^{+}(0+) = \infty$, for $t, v > 0$, let $\kappa(v) := \overline{\Pi}^{+}(\overline{\Pi}^{+,\leftarrow}(v)-) - v$, and $G_t^v := \overline{\Pi}^{+,\leftarrow}(v) Y_{t\kappa(v)}$ then, for $t > 0$, we have

$$\left({}^{(r)}X_t,\, \Delta X_t^{(r)}\right) \stackrel{\mathrm{D}}{=} \left(X_t^v + G_t^v,\, \overline{\Pi}^{+,\leftarrow}(v)\right)\Big|_{v=\mathfrak{S}_r/t}$$

(ii) if $\overline{\Pi}(0+) = \infty$, then ,for $t, v > 0$, let

$$\kappa^{\pm}(v) := \left(\overline{\Pi}\left(\overline{\Pi}^{\leftarrow}(v)-\right) - v\right) \frac{\Delta\Pi\left(\pm\overline{\Pi}^{\leftarrow}(v)\right)}{\Delta\overline{\Pi}\left(\overline{\Pi}^{\leftarrow}(v)\right)} \mathbf{1}_{\{\Delta\overline{\Pi}(\overline{\Pi}^{\leftarrow}(v))\neq 0\}}$$

$$\widetilde{G}_t^v := \overline{\Pi}^{\leftarrow}(v)(Y^+_{t\kappa^+(v)} - Y^-_{t\kappa^-(v)})$$

then, for each $t > 0$, we have

$$\left({}^{(r)}\widetilde{X}_t,\, |\widetilde{\Delta X}_t^{(r)}|\right) \stackrel{\mathrm{D}}{=} \left(\widetilde{X}_t^v + \widetilde{G}_t^v,\, \overline{\Pi}^{\leftarrow}(v)\right)\Big|_{v=\mathfrak{S}_r/t}$$

X comparable with its large jump processes

theorem suppose $\sigma^2 = 0$ and $\overline{\Pi}(0+) = \infty$. then

$$\frac{X_t}{\widetilde{\Delta X}_t^{(1)}} \xrightarrow{\mathrm{P}} 1, \text{ as } t \downarrow 0,$$

iff $\overline{\Pi}(x) \in SV$ at 0 (so that X is of bounded variation) and X has drift 0. These imply

$$(*) \qquad \frac{|\widetilde{\Delta X}_t^{(2)}|}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{\mathrm{P}} 0, \text{ as } t \downarrow 0;$$

and conversely (*) implies $\overline{\Pi}(x) \in SV$ at 0

X comparable with its large jump processes-(one-sided version)

theorem suppose $\overline{\Pi}^{+}(0+) = \infty$. then

$$\frac{X_t}{\Delta X_t^{(1)}} \xrightarrow{\mathrm{P}} 1, \text{ as } t \downarrow 0$$

iff $\overline{\Pi}^{+}(x) \in SV$ at 0, X is of bounded variation with drift 0, and $\lim_{x\downarrow 0} \overline{\Pi}^{-}(x)/\overline{\Pi}^{+}(x) = 0$.

comparing positive & negative jumps

define $\Delta X_t^+ := \max(\Delta X_t, 0)$, $\Delta X_t^- := \max(-\Delta X_t, 0)$, and

$$(\Delta X^+)_t^{(1)} := \sup_{0<s\le t} \Delta X_s^+ \quad \text{and} \quad (\Delta X^-)_t^{(1)} := \sup_{0<s\le t} \Delta X_s^-, \;\; t > 0.$$

positive and negative jump processes are independent.
when the integrals are finite, define

$$A_\pm(x) := \int_0^x \overline{\Pi}^\pm(y)\mathrm{d}y = x \int_0^1 \overline{\Pi}^\pm(xy)\mathrm{d}y.$$

theorem (comparing positive & negative jumps)

Suppose $\overline{\Pi}^{\pm}(0+) = \infty$. For (1) assume $\sum_{0<s\le t} \Delta X_s^-$ is finite a.s., and for (2) assume $\sum_{0<s\le t} \Delta X_s^+$ is finite a.s. For (3), assume both are finite a.s. Then

(1) $\dfrac{\sum_{0<s\le t} \Delta X_s^-}{\sup_{0<s\le t} \Delta X_s^+} \xrightarrow{\mathrm{P}} 0$, as $t \downarrow 0$ if and only if $\lim_{x\downarrow 0} \dfrac{\int_0^x \overline{\Pi}^-(y)\mathrm{d}y}{x\overline{\Pi}^+(x)} = 0$;

also

(2) $\dfrac{\sup_{0<s\le t} \Delta X_s^-}{\sum_{0<s\le t} \Delta X_s^+} \xrightarrow{\mathrm{P}} 0$, as $t \downarrow 0$, if and only if $\lim_{x\downarrow 0} \dfrac{x\overline{\Pi}^-(x)}{\int_0^x \overline{\Pi}^+(y)\mathrm{d}y} = 0$;

and

(3) $\dfrac{\sum_{0<s\le t} \Delta X_s^-}{\sum_{0<s\le t} \Delta X_s^+} \xrightarrow{\mathrm{P}} 0$, as $t \downarrow 0$, if and only if $\lim_{x\downarrow 0} \dfrac{\int_0^x \overline{\Pi}^-(y)\mathrm{d}y}{\int_0^x \overline{\Pi}^+(y)\mathrm{d}y} = 0$.

Finally,

$$\frac{\sup_{0<s\le t} \Delta X_s^-}{\sup_{0<s\le t} \Delta X_s^+} \xrightarrow{\mathrm{P}} 0, \text{ as } t \downarrow 0, \text{ if and only if } \lim_{x\downarrow 0} \frac{\overline{\Pi}^-(\varepsilon x)}{\overline{\Pi}^+(x)} = 0 \text{ for all } \varepsilon > 0$$

X dominating its large jump processes

versions of truncated first and second moment functions:

$$\nu(x) = \gamma - \int_{x<|y|\leq 1} y\Pi(\mathrm{d}y) \text{ and } V(x) = \sigma^2 + \int_{0<|y|\leq x} y^2\Pi(\mathrm{d}y),\ x > 0.$$

X dominating its large jump processes

versions of truncated first and second moment functions:

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variants of $\nu(x)$ and $V(x)$ are Winsorised first and second:

$$A(x) = \gamma + \overline{\Pi}^{+}(1) - \overline{\Pi}^{-}(1) - \int_x^1 (\overline{\Pi}^{+}(y) - \overline{\Pi}^{-}(y))\mathrm{d}y$$

and

$$U(x) = \sigma^2 + 2\int_0^x y\overline{\Pi}(y)\mathrm{d}y,\ \text{for } x>0.$$

$A(x)$ and $U(x)$ are continuous for $x>0$.
using Fubini's theorem,

$$A(x) = \nu(x) + x(\overline{\Pi}^{+}(x) - \overline{\Pi}^{-}(x))$$

and

$$U(x) = V(x) + x^2(\overline{\Pi}^{+}(x) + \overline{\Pi}^{-}(x)) = V(x) + x^2\overline{\Pi}(x)$$

thm (X dominating its large jump processes)

X Staying Positive Near 0 in probability. Suppose $\overline{\Pi}^{+}(0+) = \infty$.
(i) (Doney 2004) if also $\overline{\Pi}^{-}(0+) > 0$, then the following are equivalent:

$$\lim_{t \downarrow 0} P(X_t > 0) = 1; \tag{1}$$

$$\frac{X_t}{(\Delta X^{-})_t^{(1)}} \xrightarrow{\mathrm{P}} \infty, \ \text{as } t \downarrow 0; \tag{2}$$

$$\sigma^2 = 0 \quad \text{and} \quad \lim_{x \downarrow 0} \frac{A(x)}{x \overline{\Pi}^{-}(x)} = \infty; \tag{3}$$

$$\lim_{x \downarrow 0} \frac{A(x)}{\sqrt{U(x) \overline{\Pi}^{-}(x)}} = \infty; \tag{4}$$

there is a nonstochastic nondecreasing function $\ell(x) > 0$, which is slowly varying at 0, such that

$$\frac{X_t}{t \ell(t)} \xrightarrow{\mathrm{P}} \infty, \ \text{as } t \downarrow 0. \tag{5}$$

thm (X dominating its large jump processes)

(ii) Suppose X is spectrally positive, so $\overline{\Pi}^{-}(x) = 0$ for $x > 0$. Then (1) is equivalent to

$$\sigma^2 = 0 \text{ and } A(x) \geq 0 \text{ for all small } x, \tag{6}$$

and this happens if and only if X is a subordinator. Furthermore, we then have $A(x) \geq 0$, not only for small x, but for all $x > 0$.

relative stability & dominance

assume $\overline{\Pi}(0+) > 0$.

• (cf Kallenberg (2002)): relative stability (RS): $X(t)/b(t) \xrightarrow{\mathrm{P}} \pm 1$ for some measurable nonstochastic function $b(t) > 0$
iff

$$\lim_{t\downarrow 0} t\overline{\Pi}(xb_t) = 0\,,\ \lim_{t\downarrow 0} \frac{tA(xb_t)}{b_t} = \pm 1\,,\ \lim_{t\downarrow 0} \frac{tU(xb_t)}{b_t^2} = 0$$

• (Griffin & Maller (2013)): there is a measurable nonstochastic function $b_t > 0$ such that

$$\frac{|X_t|}{b_t} \xrightarrow{\mathrm{P}} 1, \text{ as } t\downarrow 0,$$

iff $X \in RS$ at 0, equivalently, iff iff $X \in RS$ at 0, equivalently, iff

$$\sigma^2 = 0 \text{ and } \lim_{x\downarrow 0} \frac{|A(x)|}{x\overline{\Pi}(x)} = \infty$$

positive relative stability & positive dominance

theorem if $\overline{\Pi}^{+}(0+) = \infty$, then the following are equivalent:

$$\frac{X_t}{(\Delta X^{+})_t^{(1)}} \xrightarrow{\mathrm{P}} \infty, \text{ as } t \downarrow 0;$$

$$\frac{X_t}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{\mathrm{P}} \infty, \text{ as } t \downarrow 0;$$

$$\sigma^2 = 0 \text{ and } \lim_{x \downarrow 0} \frac{A(x)}{x\overline{\Pi}(x)} = \infty;$$

$$X \in \mathrm{PRS} \text{ at } 0;$$

$$\lim_{x \downarrow 0} \frac{A(x)}{\sqrt{U(x)\overline{\Pi}(x)}} = \infty;$$

$$\lim_{x \downarrow 0} \frac{xA(x)}{U(x)} = \infty.$$

domain of attaction to normality

We say $X \in D(N)$ at 0 if there are $a_t \in \mathbb{R}$, $b_t > 0$, such that $(X_t - a_t)/b_t \xrightarrow{\mathrm{D}} N(0,1)$ as $t \downarrow 0$

if a_t may be taken as 0, we write $X \in D_0(N)$

- (Doney and Maller (2002)) $X \in D$ iff $\lim_{x\downarrow 0} \frac{U(x)}{x^2\overline{\Pi}(x)} = \infty$;

in fact, $D(N) = D_0(N)$ (Maller & Mason (2010));

$$X \in D \quad \text{iff} \quad X \in D_0 \quad \lim_{x\downarrow 0} \frac{U(x)}{x|A(x)| + x^2\overline{\Pi}(x)} = \infty$$

corollary[to theorem] if $\overline{\Pi}^+(0+) = \infty$, then the following are equivalent:

$$X \in D(N)$$

there is a nonstochastic function $c_t > 0$ such that $\frac{V_t}{c_t} \xrightarrow{\mathrm{P}} 1$, as $t \downarrow 0$;

$$\frac{V_t}{\sup_{0<s\le t} |\Delta X_s|^2} \xrightarrow{\mathrm{P}} \infty, \text{ as } t \downarrow 0.$$

relative stability, attraction to normality & two-sided dominance

theorem if $\overline{\Pi}(0+) = \infty$, then the following are equivalent:

$$\frac{|X_t|}{|\widetilde{\Delta X}_t^{(1)}|} \xrightarrow{\mathrm{P}} \infty, \text{ as } t \downarrow 0;$$

$$\lim_{x \downarrow 0} \frac{x|A(x)| + U(x)}{x^2 \overline{\Pi}(x)} = \infty;$$

$$\lim_{x \downarrow 0} \frac{U(x)}{x|A(x)| + x^2 \overline{\Pi}(x)} = +\infty, \quad \text{or} \quad \lim_{x \downarrow 0} \frac{|A(x)|}{x \overline{\Pi}(x)} = +\infty;$$

$$X \in D_0(N) \cup RS \text{ at } 0$$

outlook

• talk was based on B.B., Y. Fan, Y. & R.A. Maller (2016). Distributional Representations and Dominance of a Lévy Process over its Maximal Jump Processes. *Bernoulli* **22**(4), 2325–2371.

• Yuguang Fan proved NASC convergence of trimmed Levy processes in the domain of attraction of normal and stable to trimmed counterparts, including point process versions in her thesis and/or articles
—*Study in trimmed Lévy processes.* PhD thesis. ANU.
—Convergence of trimmed Lévy processes to trimmed stable random variables at 0. To appear in *Stoch. Pro. & its Appl.*.
—Tightness and Convergence of Trimmed Lévy Processes to Normality at Small Times. To appear in *J of Theo Prob*.

• current work includes small-time behaviour of extremal processes (ongoing project with Ross Maller & Ana Feirerra) & extremal processes as infinite state space Markov processes (ongoing project with Ross Maller & Sid Resnick) indexed by its order

as usual: finis delectat

Je vous remercie de votre attention!!