Recent developments in random affine and affine like recursions

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(based on the joint work with
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Random affine recursion

Let $(A_n, B_n) \in \mathbb{R} \times \mathbb{R}$ be an i.i.d. sequence. We consider the Markov chain

$$X_n = A_n X_{n-1} + B_n = (A_n, B_n) \circ X_{n-1},$$

X_0 - initial distribution. Then

$$X_n = (A_n, B_n) \circ \dots \circ (A_1, B_1) \circ X_0,$$

X_n is called a forward process. Notice that if

$$R_n = (A_1, B_1) \circ \dots \circ (A_n, B_n) \circ X_0$$

then $X_n =_d R_n$ in law. R_n is called a backward process.

Existence of a stationary distribution

Assume $\mathbb{E}\log|A_1| < 0$ and $\mathbb{E}\log^+|B_1| < \infty$.

$$\begin{aligned} R_n &= (A_1, B_1) \circ ... \circ (A_n, B_n) \circ X_0 \\ &= B_1 + A_1 B_2 + A_1 A_2 B_3 + ... + A_1..A_{n-1} B_n \end{aligned}$$

Then R_n converges a.s. to

$$R = \sum_{k=1}^{\infty} A_1 \dots A_{k-1} B_k = B_1 + A_1 \sum_{k=2}^{\infty} A_2 \dots A_{k-1} B_k = B_1 + A_1 (R \circ \theta).$$

Since $X_n =_d R_n$, the process X_n converges in distribution to R and

$$R =_d AR + B, \qquad (A, B) \perp R$$

Then ν - the law of R, is the stationary distribution of $\{X_n\}$.

Generalized Orstein-Uhlenbeck process

Bivariate Lévy process

$$(\xi, \eta) = (\xi_t, \eta_t)_{t \geq 0}$$

Generalized Orstein-Uhlenbeck process

$$V_t = e^{-\xi_t}\Big(\int_0^t e^{\xi_{s-}} \; d\eta_s + V_0 \Big),$$

V_0 the starting random variable independent of (ξ, η). For every $h > 0$, $n \in N$

$$V_{nh} =_d A_h V_{(n-1)h} + B_h,$$

where

$$(A_h, B_h) =_d \left(e^{-\xi_h}, e^{-\xi_h} \int_0^h e^{\xi_{s-}} \; d\eta_s \right)$$

Generalized Orstein-Uhlenbeck process

$$V_{nh} =_d A_h V_{(n-1)h} + B_h,$$

where

$$(A_h, B_h) =_d \left(e^{-\xi_h}, e^{-\xi_h} \int_0^h e^{\xi_{s-}} \, d\eta_s \right)$$

$$\begin{aligned} V_{nh} &= e^{-\xi_{nh}} \Big(\int_0^{nh} e^{\xi_{s-}} \, d\eta_s + V_0 \Big) \\ &= e^{-(\xi_{nh}-\xi_{(n-1)h})} e^{-\xi_{(n-1)h}} \Big(V_0 + \int_0^{(n-1)h} e^{\xi_{s-}} \, d\eta_s + \int_{(n-1)h}^{nh} e^{\xi_{s-}} \, d\eta_s \Big) \\ &= e^{-(\xi_{nh}-\xi_{(n-1)h})} V_{(n-1)h} + e^{-(\xi_{nh}-\xi_{(n-1)h})} \int_{(n-1)h}^{nh} e^{\xi_{s-}-\xi_{(n-1)h}} \, d\eta_s \end{aligned}$$

Stationary distribution for GOU

$$V_{nh} = A_h V_{(n-1)h} + B_h,$$

where

$$(A_h, B_h) = \left(e^{-\xi_h}, e^{-\xi_h} \int_0^h e^{\xi_{s-}} \, d\eta_s\right)$$

$$V = \left(\int_0^\infty e^{-\xi_{s-}} \, d\eta_s + V_0\right)$$

For every h

$$V =_d A_h V + B_h$$

Necessary and sufficient conditions for existence of V were given by A.Behme, A.Lindner and R.Maller in 2011.

Moments

More can be said if, on top of $\mathbb{E}\log|A| < 0$, we assume additionally that for some $\alpha > 0$,

$$\mathbb{E}|A|^\alpha = 1, \quad \mathbb{E}|B|^\alpha < \infty \tag{1}$$

Then for every $0 < \beta < \alpha$

$$\mathbb{E}|A|^\beta < 1$$

because $|A| \neq \mathit{const}$. In particular, for $\beta < \alpha, \beta \leq 1$

$$\mathbb{E}|R|^\beta = \mathbb{E}|\sum_{i=1}^{\infty} A_1 \ldots A_{i-1} B_i|^\beta \leq \sum_{i=1}^{\infty} (\mathbb{E}|A_1|^\beta)^{i-1} \mathbb{E}|B_i|^\beta < \infty$$

No moment of order α. $\qquad R =_d AR + B.$

Tail

$R = \sum_{k=1}^{\infty} A_1 \dots A_{k-1} B_k$ is a unique solution of

$$R =_d AR + B, \qquad (A, B) \perp R$$

Theorem (Kesten 73, Grincevicius 75, Goldie 91)
If $\mathbb{E} \log |A| < 0$, $\mathbb{E}|A|^\alpha = 1$ for some $\alpha > 0$, $0 < m_\alpha = \mathbb{E}|A|^\alpha \log |A| < \infty$, $\mathbb{E}|B|^\alpha < \infty$ and $\log |A|$ conditioned on $A \neq 0$ is non arithmetic, then

$$\mathbb{P}[R > t] \sim C_+ t^{-\alpha}, \quad \mathbb{P}[R < -t] \sim C_- t^{-\alpha}$$

and $C_+ + C_- > 0$ or R is constant. $\mathbb{P}[A < 0] > 0$ implies $C_+ = C_-$.

Later on we always assume $\mathbb{P}[Ax + B = x] < 1$, for every $x \in \mathbb{R}$ which is equivalent to $C_+ + C_- > 0$.

Positivity of C_+.

ν doesn't have atoms but it may be singular. In the case $A \geq 0$ the support of ν is $\mathbb{R} = (-\infty, \infty)$ or a half line:

- $\mathrm{supp}\nu = [c, \infty)$ and $C_+ > 0$
- $\mathrm{supp}\nu = (-\infty, c]$ and $C_- > 0$
- $\mathrm{supp}\nu = \mathbb{R}$ and $C_+, C_- > 0$

Summarize $C_+ > 0$ iff $[c, \infty) \subset \mathrm{supp}\nu$

Theorem (Guivarc'h, Le Page)

Suppose that the assumptions of the Kesten-Goldie theorem are satisfied and R is unbounded at ∞. Then there is $\epsilon > 0$ such that

$$\mathbb{P}[R > t] > \epsilon t^{-\alpha}$$

Very simple proof by Buraczewski and Damek.

Lipschitz recursions

$$\mathbb{X}_n = \Psi_n(\mathbb{X}_{n-1})$$

Ψ_n random, Lipschitz

$$\Psi(x) \geq Ax + B, \tag{2}$$

Theorem (Buraczewski, Damek)

Suppose that (A, B) satisfies assumptions of the Kesten-Goldie theorem and Ψ satisfies natural regularity assumptions. If the stationary solution $\mathbb{X}$ is unbounded at ∞ then there is $\epsilon > 0$ such that

$$\mathbb{P}[\mathbb{X} > t] > \epsilon t^{-\alpha}$$

For applications "so called Letac model"

$$\tilde{X}_n = B_n + A_n \max\{\tilde{X}_{n-1}, C_n\} = \max\{A_n\tilde{X}_{n-1} + B_n, A_nC_n + B_n\}, \quad n \geq 1.$$

is important.

Why Lipschitz

In applications we immediately go beyond $R = AR + B$
The ruin problem (of an insurance company)

$$\mathbb{P}[\sup_n \sum_{j=1}^{n} A_1 \cdots A_{j-1} B_j > t] = \mathbb{P}[M = \sup_n R_n > t] \tag{3}$$

$$M =_d \max\{AM + B, 0\} \tag{4}$$

$$\tilde{X}_n = \max\left\{A_n \tilde{X}_{n-1} + B_n, 0\right\}, \quad n \geq 1.$$

$A_n C_n + B_n = 0$

Support

$R =_d AR + B$, ν -law of R, μ - law of (A, B).

$\mathbb{E}\log A < 0$, there is $\alpha > 0$ such that $\mathbb{E}A^\alpha = 1$, $\mathbb{P}[Ax + B = x] < 1$ imply that $\mathrm{supp}\nu$ unbounded.

$\mathrm{supp}\nu$ is invariant under the action of $\mathrm{supp}\mu$, $x \in \mathrm{supp}\nu$
$(a, b) \circ x = ax + b \in \mathrm{supp}\nu$

$\mathbb{P}[A > 1] > 0$, $\mathrm{supp}\nu \neq \{x\}$
$(a, b) \in \mathrm{supp}\mu$, $a > 1$, $x \neq y$, $x, y \in \mathrm{supp}\nu$.

$|(a, b)^n \circ x - (a, b)^n \circ y| = a^n|x - y| \to \infty$

Support

μ -law of (A,B), ν -law of R, $\mathbb{P}[A=0]=0$.

$$\mathrm{supp}\nu = \overline{\left\{\frac{b}{1-a} : (a,b)\in \mathrm{supp}\bigcup_{n=1}^{\infty}\mu^{*n},\ a<1\right\}}$$

Guivarc'h, $\mathrm{supp}\nu$ does not depend on μ but only on $\mathrm{supp}\mu$.
$\frac{b}{1-a}$ is the unique fixed point of $ax+b=x$

If $x\in\mathrm{supp}\nu$, $(a,b)\in\mu$, $a>1$, $\frac{b}{1-a}<x$ then $[x,\infty)\subset\mathrm{supp}\nu$.
If $x\in\mathrm{supp}\nu$, $(a,b)\in\mu$, $a>1$, $x<\frac{b}{1-a}$ then $(-\infty,x]\subset\mathrm{supp}\nu$.

Invariance of the support of ν allows to generate a lot of points see the book by Buraczewski, Damek, Mikosch *Stochastic Models with Power-Law Tails. The Equation $X=AX+B$*
$\mathrm{supp}\nu=\mathbb{R}$ or a half line

Support

$(a_1, b_1), (a_2, b_2) \in \mathrm{supp}\mu,\ a_1 > 1, a_2 < 1,\ \mu$ -law of (A, B) If

$$\frac{b_1}{1-a_1} < \frac{b_2}{1-a_2}$$

then

$$\Big[\frac{b_2}{1-a_2}, \infty\Big) \subset \mathrm{supp}\nu$$

If $\mathbb{P}[A = 1, B > 0] > 0$ then

$$\Big[\frac{b_2}{1-a_2}, \infty\Big) \subset \mathrm{supp}\nu$$

If $\mathbb{P}[A = 1, B > 0] = 0$ for every $(a_1, b_1), (a_2, b_2) \in \mathrm{supp}\mu$, $a_1 > 1, a_2 < 1$

$$\frac{b_1}{1-a_1} > \frac{b_2}{1-a_2}$$

then

$$\mathrm{supp}\nu = (-\infty, c]$$

Letac recursion

$$\tilde{X}_n = B_n + A_n \max\{\tilde{X}_{n-1}, C_n\} = \max\{A_n\tilde{X}_{n-1} + B_n, A_nC_n + B_n\}, \quad n \geq 1.$$

$$X_n' = \max\{A_nX_{n-1}' + B_n, 0\}, \quad n \geq 1.$$

$$\tilde{X}_n \geq X_n, \quad X_n' \geq X_n. \tag{5}$$

Under assumptions of Goldie-Kesten Theorem plus $\mathbb{E}[A^\alpha |C|^\alpha] < \infty$, Goldie proved that

$$\mathbb{P}[\tilde{X} > t] \sim C_L t^{-\alpha} \qquad \text{as } t \to \infty,$$

but no characterization of positivity of C_L. Some sufficient conditions in Goldie, and in Collamore, Vidyashankar.

Letac recursion

$$\tilde{X}_n = B_n + A_n \max\{\tilde{X}_{n-1}, C_n\}$$

Theorem (Buraczewski, Damek)

Suppose that the assumptions of the Kesten-Goldie theorem are satisfied and $\tilde{X}$ (X') is unbounded at ∞. Then there is $\epsilon > 0$ such that

$$\mathbb{P}[\tilde{X} > t] > \epsilon t^{-\alpha}$$

$\tilde{X}$ is unbounded at ∞ if either $\mathbb{P}[A = 1, B > 0] > 0$ or $\mathbb{P}[A = 1, B > 0] = 0$ and $N_3 < \max\{N_1, N_2\}$.

$$(A, B, C) \asymp \mu$$

$$\begin{aligned} N_1 &= \sup\{ac + b : \ (a, b, c) \in \operatorname{supp} \mu\}, \\ N_2 &= \sup\{b(1-a)^{-1} : \ (a, b, c) \in \operatorname{supp} \mu \text{ and } a < 1\}, \\ N_3 &= \inf\{b(1-a)^{-1} : \ (a, b, c) \in \operatorname{supp} \mu \text{ and } a > 1\}. \end{aligned} \tag{6}$$

Lipschitz recursions

$$\mathbb{X}_n = \Psi_n(\mathbb{X}_{n-1})$$

Ψ_n random, Lipschitz, $\mathbb{E} \log Lip(\Psi) < 0$

$$Ax - B \leq \Psi(x) \leq Ax + B, \tag{7}$$

Theorem (Buraczewski, Damek, Mirek)

Suppose that (A, B) satisfies assumptions of the Kesten-Goldie theorem and Ψ satisfies natural regularity assumptions. Then for the stationary solution $\mathbb{X}$ we have (Mirek, 2011)

$$\lim_{t\to\infty} t^\alpha \mathbb{P}(\mathbb{X} > t) = C_+,$$
$$\lim_{t\to\infty} t^\alpha \mathbb{P}(\mathbb{X} < -t) = C_-.$$

If $\mathbb{X}$ is unbounded at ∞ then $C_+ > 0$ (Buraczewski, Damek).

Second recursion

$$\begin{aligned} X_{n+1} &= A_{n+1} X_n + B_n \\ \tilde{X}_{n+1} &= A_{n+1} \tilde{X}_n + \tilde{B}_n \\ \tilde{B}_{n+1} &= \min(-1, B_n) \\ X_0 &= \tilde{X}_0 = 0 \end{aligned}$$

$$\begin{aligned} \mathrm{supp}\tilde{R} &\subset (-\infty, 0] \\ \mathbb{P}[\tilde{R} < t] t^{\alpha} &\to \tilde{C}_- > 0 \\ R_n &\geq \tilde{R}_n \geq \tilde{R} \end{aligned}$$

$$\sum_{j=1}^{n} A_1 \cdots A_{j-1} B_j \geq \sum_{j=1}^{n} A_1 \cdots A_{j-1} \tilde{B}_j \geq \sum_{j=1}^{\infty} A_1 \cdots A_{j-1} \tilde{B}_j$$

A lemma

$$M = \max_n \Pi_n, \ \ \Pi_n = A_1 \cdots A_n$$

$$\lim_{t\to\infty} \mathbb{P}[M > t]t^\alpha = c_1 > 0.$$

Lemma

Let $U_n = \{\Pi_n > t, \tilde{R}_n > -Ct\}$. Then there is $D > 0$ such that

$$\mathbb{P}[\bigcup_n U_n] \geq \frac{c_1}{4} t^{-\alpha}.$$

Not that surprising because

$$\mathbb{P}[\tilde{R}_n \leq -Ct] \leq \mathbb{P}[\tilde{R} \leq -Ct] \leq \tilde{C}_- C^{-\alpha} t^{-\alpha}$$

Proof

$$\mathbb{P}\Big[\bigcup_n \{\Pi_n > t, R_n > -Ct\}\Big] \geq \frac{c_1}{4} t^{-\alpha}$$

$$\mathbb{P}[R \circ \theta^n > C+1] = \eta > 0, \quad R \text{ unbounded}$$

$$\begin{aligned} &\mathbb{P}[\Pi_n > t, R_n > -Ct]\mathbb{P}[R \circ \theta^n > C+1] \\ = &\mathbb{P}[\Pi_n > t, R_n > -Ct, R \circ \theta^n > C+1] \end{aligned}$$

$$R = R_n + \Pi_n R(\theta^n \omega) > -Ct + (C+1)t = t$$

Positivity of C_+.

In the case $A \geq 0$ the support of ν is $\mathbb{R} = (-\infty, \infty)$ or a half line: $C_+ > 0$ iff $[c, \infty) \subset \mathrm{supp}\nu$

Theorem (Guivarc'h, Le Page)

Suppose that the assumptions of the Kesten-Goldie theorem are satisfied: $\mathbb{E} \log |A| < 0$, $\mathbb{E}|A|^\alpha = 1$ *for some* $\alpha > 0$, $0 < m_\alpha = \mathbb{E}|A|^\alpha \log |A| < \infty$, $\mathbb{E}|B|^\alpha < \infty$ *and* R *is unbounded at* ∞*. Then there is* $\epsilon > 0$ *such that*

$$\mathbb{P}[R > t] > \epsilon t^{-\alpha}$$

Proof

$$\frac{c_1}{2} t^{-\alpha} \leq \mathbb{P}[M > t] = \mathbb{P}(\bigcup_n \{\Pi_n > t\})$$

$$= \mathbb{P}(\bigcup_n \{\Pi_n > t \text{ and } \tilde{R} \leq -Ct\})$$

$$+ \mathbb{P}(\bigcup_n \{\Pi_n > t \text{ and } \tilde{R} > -Ct\})$$

$$\leq \mathbb{P}[\tilde{R} \leq -Ct] + \mathbb{P}(\bigcup_n \{\Pi_n > t \text{ and } \tilde{R} > -Ct\})$$

$$\leq \frac{2\tilde{C}_-}{C^\alpha} t^{-\alpha} + \mathbb{P}(\bigcup_n \{\Pi_n > t \text{ and } \tilde{R}_n > -Ct\})$$

$$\leq \frac{2\tilde{C}_-}{C^\alpha} t^{-\alpha} + \mathbb{P}[\bigcup_n U_n]$$

Identity

$$
\begin{aligned}
R &= \sum_{j=1}^{\infty} A_1 \cdots A_{j-1} B_j \\
& \sum_{j=1}^{\infty} A_1 \cdots A_{j-1} B_j + A_1 \cdots A_n \sum_{j=n+1}^{\infty} A_{n+1} \cdots A_{j-1} B_j \\
&= R_n + \Pi_n (R \circ \theta^n)
\end{aligned}
$$

Proof

$$\mathbb{P}[R > C+1] = \eta > 0 \quad R \text{ unbounded}$$

$$\begin{aligned}
\frac{c_1}{4} t^{-\alpha}\eta \leq & \eta\mathbb{P}[\bigcup_n U_n] \\
& \eta\mathbb{P}[\bigcup_n U_n \cap (\bigcup_{k=1}^{n-1} U_k)^c] \text{ disjoint} \\
& = \sum_{n=1}^{\infty} \mathbb{P}[\bigcup_n U_n \cap (\bigcup_{k=1}^{n-1} U_k)^c]\mathbb{P}[R(\theta^n\omega) > C+1] \\
& = \sum_{n=1}^{\infty} \mathbb{P}[\bigcup_n U_n \cap (\bigcup_{k=1}^{n-1} U_k)^c \cap \{R(\theta^n\omega) > C+1\}] \\
& \leq \mathbb{P}(R > t)
\end{aligned}$$

Proof

$$U_n \cap (\bigcup_{k=1}^{n-1} U_k)^c \cap \{R(\theta^n \omega) > C+1\}$$

$$\Pi_n \geq t, \tilde{R}_n > -Ct, R(\theta^n \omega) > C+1$$

$$R_n > -Ct$$

$$R = R_n + \Pi_n R(\theta^n \omega) > -Ct + (C+1)t = t$$

$$U_n \cap (\bigcup_{k=1}^{n-1} U_k)^c \cap \{R(\theta^n \omega) > C+1\}$$

$$\subset \{R > t\} \cap U_n \cap (\bigcup_{k=1}^{n-1} U_k)^c \text{ disjoint}$$