Totally **O**rdered **M**easured Trees and Splitting Trees with Infinite Variation

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Work in progress in collaboration with
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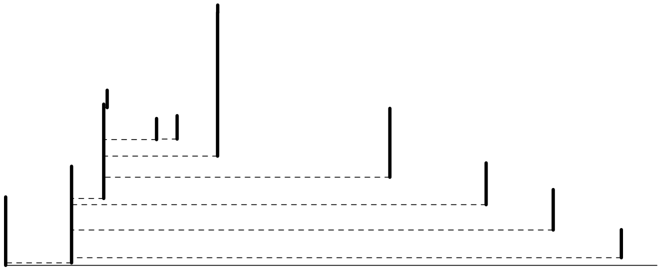
Lévy Processes, Angers, Jul-25-2016

Motivation 1: A chronological model on trees



The Model

Consider a population in which individuals live for a certain lifetime during which they reproduce at constant rate b. At each reproduction event only one offspring is produced. Assume that that the lifetimes of different individuals are independent and with law Λ.

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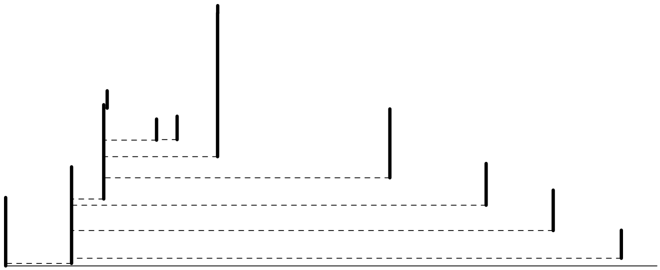

Three associated branching processes

- $Z^c = (Z^c_t, t \geq 0)$: Z^c_t is the number of individuals alive at time t
- $Z^d = \left(Z^d_n, n \in \mathbb{N}\right)$: Z^d_n size of the n-th generation.
- $Z^J = \left(Z^J_n, n \in \mathbb{N}\right)$: Sum of lifetimes of individuals at generation n.

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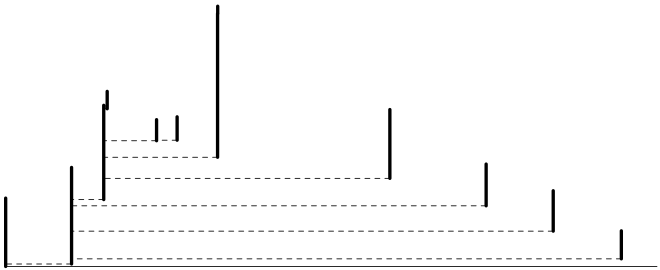

Extinction (or finitude) criteria

The genealogical or chronological trees are finite almost surely if and only if

$$m = b \int_0^\infty r\Lambda(dr) \leq 1.$$

$$\begin{cases} m < 1 & \text{subcritical} \\ m = 1 & \text{critical} \\ m > 1 & \text{supercritical} \end{cases}$$

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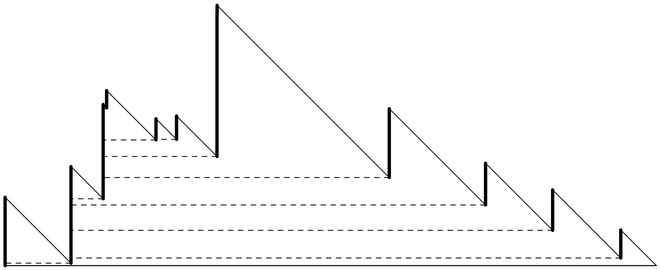

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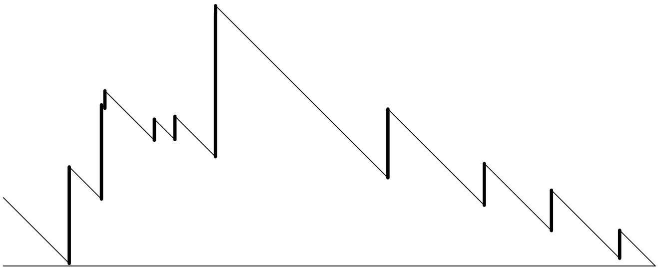

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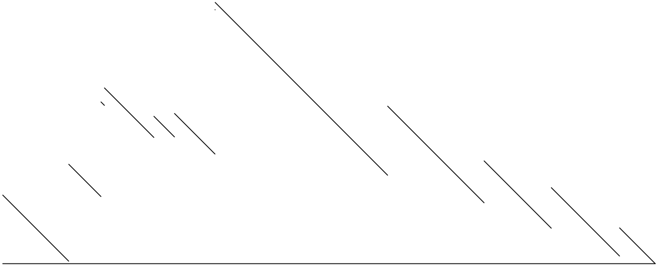

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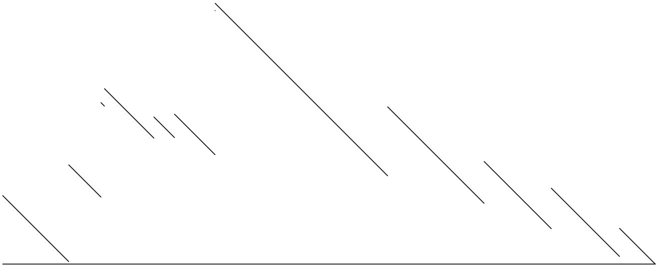
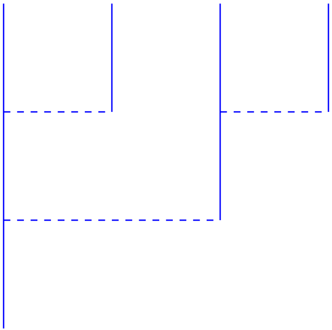


Theorem, Lambert (2010)

In the (sub)critical case, the contour of a Splitting Tree is a (spectrally positive) compound Poisson process with random initial state and jump distribution with law Λ, a jump rate of b, stopped upon reaching zero.

Amaury Lambert, *The contour of splitting trees is a Lévy process*, Ann. Probab. **38** (2010), no. 1, 348–395. MR: 2599603

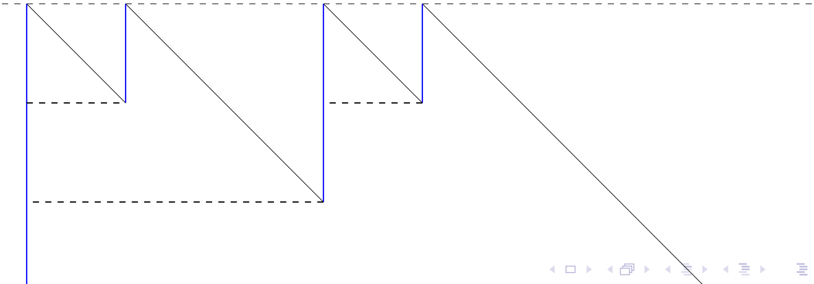
Motivation 1: A chronological model on trees



Theorem, Lambert (2010)

In the supercritical case, the contour of a Splitting Tree truncated at height r is a (spectrally positive) compound Poisson process with random initial state and jump distribution with law Λ, a jump rate of b, **reflected under** r and stopped upon reaching zero.

Amaury Lambert, *The contour of splitting trees is a Lévy process*, Ann. Probab. **38** (2010), no. 1, 348–395. MR: 2599603

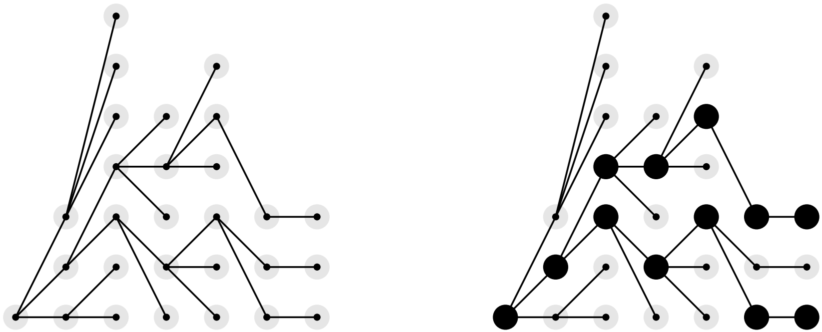


Motivation 1: A chronological model on trees

First motivation

- How can one describe the chronology in the case of continuum trees?
- What would be the link between a continuum splitting tree model and Lévy processes?
- Can one make sense and describe the associated genealogy?

Motivation 2: Supercritical models of continuum trees



The population has individuals of two types:

Prolific type Those who have descendants on every generation

Non-prolific type Those who don't.

Motivation 2: Supercritical models of continuum trees

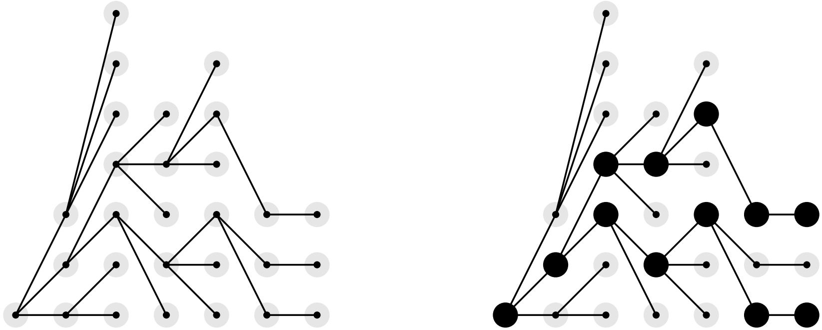
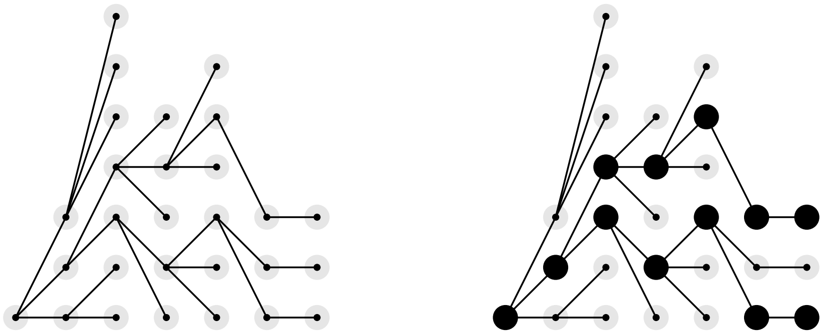


Figure: Left: the first 7 generations of an infinite plane tree. Generations increase from left to right. On each generation, labels (lexicographically) increase from bottom to top. (Hence, the tree is $\{\emptyset, 1, 2, 3, 11, 12, 21, 22, 31, 32, 33, \ldots\}$). Right: the prolific individuals are identified by black circles. Notice that the root has three subtrees above it: two finite ones and an infinite one.

Motivation 2: Supercritical models of continuum trees

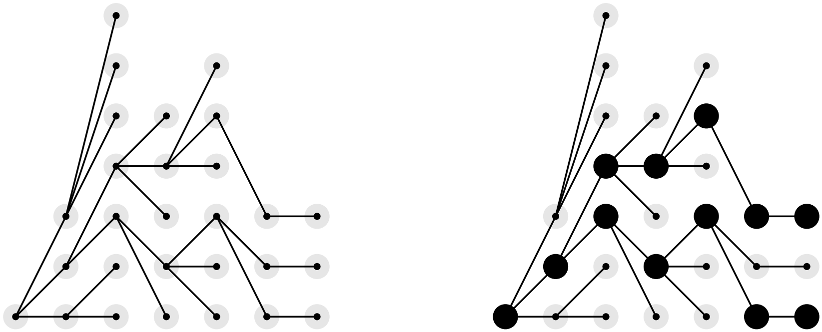


Let $(Z_n^1, n \geq 0)$ be the quantity of prolific individuals in generation n of a supercritical Galton-Watson tree.
Let $(Z_n^2, n \geq 0)$ be the quantity of non-prolific individuals at generation n.

Classical Result

The process $\big((Z_n^1, Z_n^2), n \geq 0\big)$ is a two-type branching process.

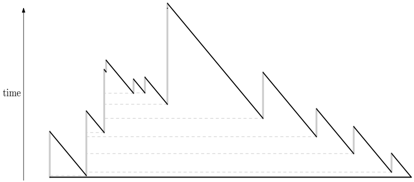
Motivation 2: Supercritical models of continuum trees

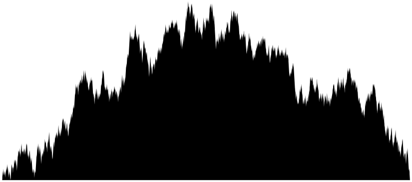


Motivation 2:

- Is there a continuum tree analogue of a two-type decomposition in a setting related to Lévy trees?

Preliminaries: The tree coded by a càdlàg function





Let $x : [0, \zeta] \to [0, \infty)$ such that $x_\zeta = 0$ and $\Delta x_t = x_t - x_{t-} \geq 0$.
Define

$$d_x(t_1, t_2) = x_{t_1} + x_{t_2} - 2 \min_{t \in [t_1, t_2]} x_t.$$

Fact: d_x is a pseudo-distance.
Let $\sim_x$ be the equivalence relation defined by

$$t_1 \sim_x t_2 \text{ if and only if } d_x(t_1, t_2) = 0$$

and define τ_x to be the set of equivalence classes. ρ_x is the class of ζ.
Fact: (τ_x, d_x, ρ_x) is a compact rooted real tree.

Jean-François Le Gall, *Random real trees*, Ann. Fac. Sci. Toulouse Math. (6) **15** (2006), no. 1, 35–62. MR: 2225746
Thomas Duquesne, *The coding of compact real trees by real valued functions*, http://arxiv.org/abs/math/0604106, 2008

Real trees

Definition

An $\mathbb{R}$-**tree** (or **real tree**) is a metric space (τ, d) satisfying the following properties:

Completeness (τ, d) is complete.

Existence of geodesics For all $\sigma_1, \sigma_2 \in \tau$ there exists a unique isometric embedding

$$\phi_{\sigma_1,\sigma_2} : [0, d(\sigma_1, \sigma_2)] \to \tau$$

such that $\phi(0) = \sigma_1$ and $\phi(d(\sigma_1, \sigma_2)) = \sigma_2$.

Lack of loops For every injective continuous mapping $\phi : [0, 1] \to \tau$ such that $\phi(0) = \sigma_1$ and $\phi(1) = \sigma_2$, the image of $[0, 1]$ under ϕ equals the image of $[0, d(\sigma_1, \sigma_2)]$ under ϕ_{σ_1,σ_2}.

A triple (τ, d, ρ) consisting of a real tree (τ, d) and a distinguished element $\rho \in \tau$ is called a **rooted tree**.

A. W. M. Dress and W. F. Terhalle, *The real tree*, Adv. Math. **120** (1996), no. 2, 283–301. MR: 1397084
Steven N. Evans, Jim Pitman, and Anita Winter, *Rayleigh processes, real trees, and root growth with re-grafting*, Probab. Theory Related Fields **134** (2006), no. 1, 81–126. MR: 2221786

Genealogical partial order

If (τ, d, ρ) is a rooted tree and $\sigma_1, \sigma_2 \in \rho$, we define the **closed interval** $[\sigma_1, \sigma_2]$ to be the image of $[0, d(\sigma_1, \sigma_2)]$ under ϕ_{σ_1,σ_2}.

We can now define the **genealogical partial order** $\preceq$ by stating that

$$\sigma_1 \preceq \sigma_2 \quad \text{if and only if} \quad \sigma_1 \in [\rho, \sigma_2].$$

Because a tree has no loops, there is a unique element, denoted $\sigma_1 \wedge \sigma_2$ such that

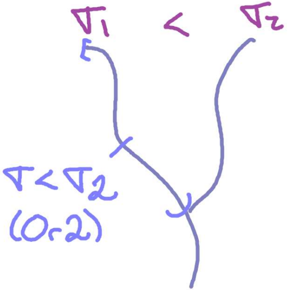
$$[\rho, \sigma_1] \cap [\rho, \sigma_2] = [\rho, \sigma_1 \wedge \sigma_2].$$

Totally Ordered Measured (TOM) trees

Definition

A **real tree** (τ, d, ρ) is called **totally ordered** if there exists a total order $\leq$ on τ which satisfies

- Or1 $\sigma_1 \preceq \sigma_2$ implies $\sigma_2 \leq \sigma_1$ and
- Or2 $\sigma_1 < \sigma_2$ implies $[\sigma_1, \sigma_1 \wedge \sigma_2) < \sigma_2$.



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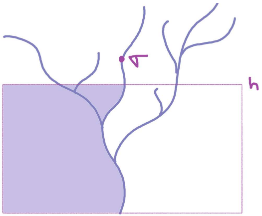
A totally ordered real tree is called **measured** if there exists a measure μ on the Borel sets of τ satisfying:

- Mes1 for every σ (that is not the $\leq$-first element of τ) and every $h > 0$,

$$\mu(\{\tilde{\sigma} \leq \sigma : d(\rho, \tilde{\sigma}) \leq h + d(\rho, \sigma)\}) \in (0, \infty).$$



- Mes2 μ is diffuse.

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- Mes2 μ is diffuse.

$$\mathbf{c} = ((\tau, d, \rho), \leq, \mu).$$

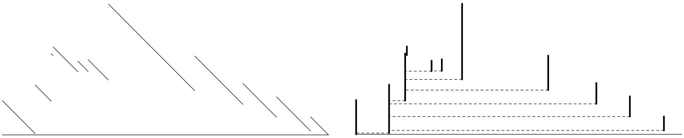
We will be exclusively interested in locally compact TOM trees.
In the compact case, μ is finite.

The exploration process and the contour

Theorem

Let $\mathbf{c}$ be a compact TOM tree and let $m = \mu(\tau)$.
There is a unique càdlàg function $f_{\mathbf{c}} : [0, m] \to [0, \infty)$ with no negative jumps such that the tree coded by $f_{\mathbf{c}}$ is isomorphic to $\mathbf{c}$.

The function $f_{\mathbf{c}}$ is called the **contour** of the tree.
The function that sends t to $[t]_f$ is called the **exploration** process.



(a) (b)

Amaury Lambert and Gerónimo Uribe Bravo, *Totally ordered measured trees and splitting trees with infinite variation*, arXiv:1607.02114

Topological remarks on the set(!) of compact TOM trees

Distance between compact TOM trees $\mathbf{c}_1$ and $\mathbf{c}_2$ in terms of the distance of their contours f_1 and f_2:
Suppose that the supports of f_i is $[0, m_i]$ and that $m_1 < m_2$, say.
First extend f_1 to $[0, m_2]$ by declaring it constant on $[m_1, m_2]$. We then define

$$d(\mathbf{c_1}, \mathbf{c}_2) = d_{m_2}(f_1, f_2) + |m_2 - m_1| \,,$$

where d_{m_2} is the Skorohod J_1 distance on $[0, m_2]$ defined as

$$d_{m_2}(f_1, f_2) = \sup_{\lambda} \sup_{s \le t} |f_1(s) - f_2 \circ \lambda(s)|$$

where λ runs over all strictly increasing continuous functions of $[0, m_2]$ into itself.

Topological remarks on the set(!) of compact TOM trees

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$$\mathbf{c_n} = ((\tau_n, d_n, \rho_n), \leq_n, \mu_n) \to \mathbf{c} \text{ implies } ((\tau_n, d_n, \rho_n), \mu_n) \to ((\tau, d, \rho), \mu)$$

in the Gromov-Hausdorff-Prokhorod topology:

$$d_{GHP}(((\tau_1, d_1, \rho_1), \mu_1), ((\tau_n, d_n, \rho_n), \mu_n))$$
$$= \inf_{\phi_1, \phi_2, \tau} \left[d_H^{\tau}(\phi_1(\tau_1), \phi_2(\tau_2)) + d^{\tau}(\phi_1(\rho_1), \phi_2(\rho_2)) + d_P^{\tau}(\mu_1 \circ \phi_1^{-1}, \mu_2 \circ \phi_2^{-1}) \right]$$

Compact TOM trees with the splitting property

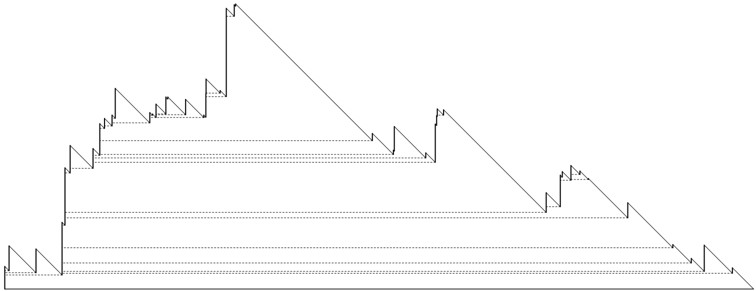
Let $X = (X_t, t \geq 0)$ be a Lévy process with no negative jumps (spLp) and Laplace exponent Ψ:

$$\mathbb{E}\left(e^{-uX_t}\right) = e^{-t\Psi(u)}.$$

Recall that if $\underline{X}_t = \min_{s \leq t} X_s$ then $X - \underline{X}$ is a strong Markov process. If

$$\Psi'(0+) = -\mathbb{E}(X_1) \geq 0$$

then $X_t \to -\infty$ as $t \to \infty$ or X oscilates and so $X - \underline{X}$ is recurrent.
Let n be the law of excursions of $X - \underline{X}$ away from zero.
Let $\mathbf{C} = ((\tau_X, d_X, \rho_X), \leq, \mu)$ be the tree coded by X under n.



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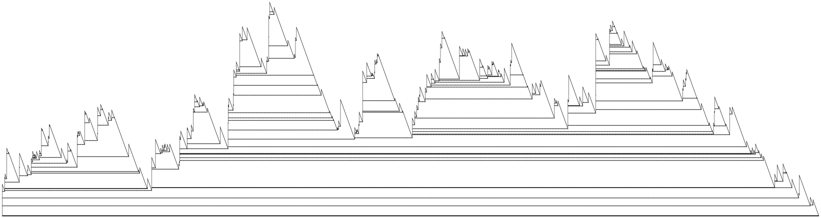

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Let n be the law of excursions of $X - \underline{X}$ away from zero.
Let $\mathbf{C} = ((\tau_X, d_X, \rho_X), \leq, \mu)$ be the tree coded by X under n.

$\mathbf{C}$ has **sojourn a** if $\mu([\rho, \sigma]) = a\, d(\rho, \sigma)$.

Theorem

Under n, $\mathbf{C}$ has the following splitting property: for any $t > 0$ and on the set $\{\mu(\tau_X)\} > t$, the trees coming off to the right of $[\rho, \Phi(t)]$ fall according to a Poisson random measure with intensity Leb $\times$ n. Also, almost surely, $\mathbf{C}$ has sojourn $a = \lim_{\lambda \to \infty} \lambda / \Psi(\lambda)$.
Conversely, if n is a σ-finite measure on compact TOM trees with the splitting property, concentrated on binary trees with sojourn a, the law of the contour is the excursion law of a Lévy process which does not drift to infinity above its cumulative minimum process.

Construction of locally compact TOM trees

Truncation

Fix a truncation level r.
τ goes to
$\tau^r = \{\sigma \in \tau : d(\sigma, \rho) \leq r\}$.
$f_r = f \circ C_r$ where C^r is the inverse of $\int_0^{\cdot} \mathbf{1}_{f(s) \leq r}\, ds$.

Compatibility under truncation

A sequence (τ_n) is compatible under pruning if for some levels $r_n \to \infty$, $\tau_n = \tau_{n+1}^{r_n}$.

Proposition

There exists a unique locally compact TOM tree τ such that $\tau^{r_n} = \tau_n$.

Pruning

Select $B \subset \tau$ and let

$$\tilde{B} = \{\sigma \in \tau : \sigma \preceq \tilde{\sigma} \text{ for some } \tilde{\sigma} \in B\}\,.$$

Let $\tau^B = \tau \setminus \tilde{B}$.
$f^B = f \circ C^B$ where C^B is the inverse of Leb $\left\{ s \leq \cdot : [s] \in \tilde{B}_n \right\}$.

Compatibility under pruning

A sequence (τ_n) is compatible under pruning if there exists $B_n \subset \tau_{n+1}$ such that $\tau_{n+1}^{B_n} = \tau_n$.

Proposition

If $\inf\{d(\sigma, \rho) : \sigma \in B_n\} \to \infty$, there exists a unique locally compact TOM tree τ such that every τ_n is the pruning of τ.

Locally Compact Splitting trees

Let $X = (X_t, t \geq 0)$ be a supercritical (possibly killed) Lévy process with no negative jumps (spLp) and Laplace exponent Ψ

$$\mathbb{E}\left(e^{-uX_t}\right) = e^{-t\Psi(u)} \quad b = \sup\{\lambda \geq 0 : \Psi(\lambda) = 0\} \quad \text{supercritical: } \mathbf{b} > \mathbf{0}.$$

$$\Psi^{\#}(\lambda) = \Psi(\lambda + b)$$

Measure on excursions n Excursion measure of X above its cumulative minimum process:

$$n = n^{\#} + b\mathbb{P}^{\rightarrow}$$

Measure on bounded and finite excursions n^r

$$n^r = n^{\#,r} + b\mathbb{Q}^{\rightarrow,r}$$

where $q^{\rightarrow,r}$ is the concatenation of $\mathbb{P}^{\rightarrow,r}$ and independent copies of $\mathbb{P}^r_r$ until one reaches zero.

Measure on compact TOM trees η^r Law of tree under n^r.

Proposition: $(\eta^r, r \geq 0)$ are consistent under truncation.

Locally Compact Splitting trees

Theorem

There exists a unique measure η^{Ψ} on locally compact TOM trees whose truncation at level r equals η^{r}.

The measure η^{Ψ} satisfies the splitting property, is concentrated on binary trees, has constant sojourn

$$a = \lim_{\lambda \to \infty} \frac{\lambda}{\Psi(\lambda)},$$

and assigns finite measure to non-compact TOM trees. η^{Ψ} charges non-compact trees if and only if Ψ is supercritical.

Conversely, if a non-zero measure κ on locally compact TOM trees has the splitting property, is concentrated on binary trees and there exists $a \geq 0$ such that under κ the tree has sojourn a almost everywhere, then there exists a spectrally positive Lévy process with Laplace exponent Ψ such that $\kappa = \eta^{\Psi}$.

Infinite lines of descent

Infinite line of descent

An infinite line of descent is an isometry from $[0, \infty)$ into a real tree τ.

Proposition

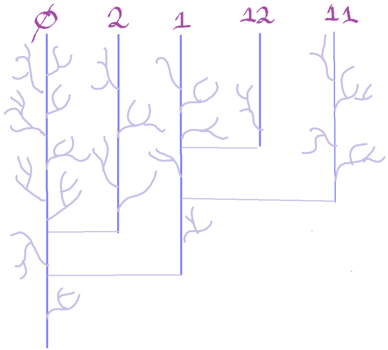
On a locally compact TOM tree τ, let

$$\mathscr{I} = \{\sigma \in \tau : \sigma \text{ has an infinite line of descent}\}.$$

Then: $\mathscr{I} = \emptyset$ if and only if τ is compact.

Otherwise: there exists a plane tree τ_I and a collection of infinite lines of descent $(I_u : u \in \tau_I)$ such that

1. $\bigcup_{u \in \tau_I} I_u([0, \infty)) = \mathscr{I}$
2. $I_u \cap I_v = \emptyset$ unless $u = \pi(v)$, in which case the only common point is $I_v(0)$.



Yule tree decomposition

Theorem

Let Ψ be the Laplace exponent of a spLp drifting to ∞.
Let b be the largest root of Ψ and define $\Psi^{\#}(\lambda) = \Psi(\lambda + b)$.
Let Υ be the law of the projective limit of: the concatenation of a Ψ-Lévy process conditioned to stay positive time-changed to remain below r followed by a $\Psi^{\#}$-Lévy process started at r, time-changed to remain below r and killed when it reaches zero.
Then the law η^{Ψ} of the splitting tree associated to Ψ decomposes as:

$$\eta^{\Psi} = \eta^{\Psi^{\#}} + b\Upsilon_{\text{tree}}.$$

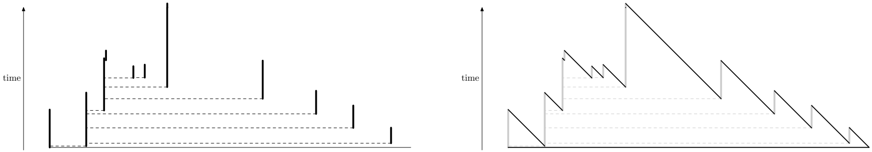
The measure Υ_{tree} is obtained recursively from the law Υ:

- On a tree $T_{\emptyset}$ with law Υ, graft iid trees $T_1, T_2, \ldots,$ (law Υ) at rate b
- Repeat the procedure on each of the grafted trees.... and so on.

Corollary

The tree of infinite lines of descent is a Yule tree.

The genealogy of a splitting tree



$$H_t^n = \text{ generation of the individual visited at time } t.$$

Codes the genealogy associated to the chronological tree.
For general Lévy processes X, let Z stand for X or for X^r:

$$H_t = \lim_{\varepsilon \to 0} \frac{\#\text{Trees to the left of } [0, \phi(t)] \text{ of measure } > \varepsilon}{n(\text{Measure} > \varepsilon)}$$

Theorem (Duquesne-Le Gall)

For subcritical Lévy processes satisfying Grey's condition $\int^\infty 1/\Psi(u)\, du < \infty$, the height process H admits a continuous modification.

Thomas Duquesne and Jean-François Le Gall, *Random trees, Lévy processes and spatial branching processes*, Astérisque (2002), no. 281, vi+147. MR: 1954248

The genealogy of a splitting tree

Theorem

For any Lévy process satisfying Grey's condition $\int^{\infty} 1/\Psi(u)\, du < \infty$, the height process H under n^r admits a continuous modification. These height processes code trees compatible under pruning.

The genealogy of a splitting tree

$$Z_a^1 = \#\left\{\sigma \in \Gamma : \sigma \text{ has an infinite line of descent and } d(\sigma,\rho) = a\right\}.$$

Theorem

Under γ^{lc}, the process Z^1 is a non-decreasing branching process with values in $\mathbb{N}$ and jumps in $\{1, 2, , \ldots\}$ which starts at 1.

Jump rate b

jump distribution determined by

$$\mathbb{P}\big(\Delta Z_{T_1}^1 = k\big) = \frac{1}{\Phi^{\uparrow}(b)}\left[\mathbf{1}_{k=1}\beta b + \int_0^\infty \frac{(bx)^{k+1}}{(k+1)!} e^{-bx}\,\pi(dx)\right]$$

$$\Phi^{\uparrow}(\beta) = \beta b + \int_0^\infty (1 - e^{-by} - bye^{-by})\,\pi(dy)\,.$$

$$A \mapsto \mu(\{\sigma \in \tau : d(\sigma,\rho) \in A\})$$

admits a càdlàg density Z^2. Finally, the process $Z = \left(Z^1, Z^2\right)$ is a two-type branching process with values in $\mathbb{N} \times [0,\infty)$ started at $(1,0)$.

On the two-dimensional branching process

Z^2 does not influence the behavior of Z^1

$$X^1 = \left(X^{1,1}, X^{1,2}\right) \quad \text{and} \quad X^2$$

(with values in $\mathbb{N} \times [0, \infty)$ and $\mathbb{R}$ respectively)

$$Z_t^1 = 1 + X^{1,1}_{\int_0^t Z_s^1\, ds} \qquad\qquad Z_t^2 = X^2_{\int_0^t Z_s^2\, ds} + X^{1,2}_{\int_0^t Z_s^1\, ds}.$$

if Z starts at (k, x) then,

$$Z_t \approx X_t^{k,x} = (k + X^{1,1}_{kt}, x + X^2_{xt} + X^{1,1}_{kt}) \quad t \to 0.$$

$$\Psi^1(\lambda_1, \lambda_2) = -\log \mathbb{E}\left(e^{-\lambda_1 X_1^{1,1} - \lambda_2 X_1^{1,2}}\right) \quad \text{and} \quad \Psi^2(\lambda) \quad = -\log \mathbb{E}\left(e^{-\lambda X_t^2}\right)$$

$\Psi^2 = \Psi^{\#}$

X^1 has drift coefficient $(0, b)$

$\pi^1(dx, dk) = \beta\, \delta_{(1,0)}(dk, dx) + \sum_{l=0}^{\infty} \delta_l(dk)\, \pi(dx)\, e^{-bx} \frac{(bx)^{l+1}}{(l+1)!}$.

M. Emilia Caballero, José Luis Pérez Garmendia, and Gerónimo Uribe Bravo, *Affine processes on $\mathbb{R}_+^m \times \mathbb{R}^n$ and multiparameter time changes*, arXiv e-prints (2015), To appear in Ann. Inst. H. Poincaré