On suprema of Lévy processes

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Joint work with L.Chaumont, M. Kwaśnicki, M. Ryznar and G. Serafin.

Introduction

- Let $(X_t)_{t\geq 0}$ be a one-dimensional Lévy process, i.e. a real-valued process with independent stationary increments starting from 0 and having càdlàg trajectories.
- $\Psi(\xi)$ - Lévy-Khintchin exponent of X_t, i.e. we have

$$\mathbf{E}e^{i\xi X_t} = e^{-t\Psi(\xi)}, \quad t \geq 0, \xi \in \mathbb{R}.$$

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- Supremum functional

$$\overline{X_t} = \sup_{0\leq s\leq t} X_s$$

and its density function (if it exists)

$$f_t(x) = \frac{\mathbf{P}(\overline{X}_t \in dx)}{dx}, \quad x > 0, t > 0.$$

Aim: Describe $\mathbf{P}(\overline{X_t} < x)$ and $f_t(x)$ by providing formulas and/or sharp estimates and/or asymptotics.

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- In general, the double Laplace transform of $\overline{X_t}$ is known (Baxter and Donsker 1957). For symmetric Lévy process we have

$$\int_0^\infty e^{-zt}\mathbf{E}e^{-\xi\overline{X_t}}dt = \frac{1}{\sqrt{z}}\exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi\log(z+\Psi(\eta))}{\xi^2+\eta^2}d\eta\right).$$

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- In some special cases, there exist *explicit* formulas for $\overline{X_t}$
  - $\Psi(\xi) = \xi^2$, Brownian motion,
  - $\Psi(\xi) = \xi$, symmetric Cauchy process,
  - $\Psi(\xi) = 1 - \cos\xi$, symmetric compound Poisson process,
  - $\Psi(\xi) = -i\xi\alpha + \lambda(e^{i\xi} - 1)$, Poisson process with drift,
  - Double series representation for stable processes.

- We denote by $\kappa(z,\xi)$ the Laplace exponent of the bivariate subordinator (τ_s, H_s).

$$\kappa(z,\xi) = \exp\left(\int_0^\infty \int_{[0,\infty)} (e^{-t} - e^{-zt-\xi x}) t^{-1} \mathbf{P}(X_t \in dx)\right)$$

and

$$\kappa(z,0) = \mathtt{d}z + \int_{(0,\infty)} (1 - e^{-zx}) \pi(dx)$$

where $\mathtt{d}$ and π are a drift and a Levy measure of ladder-time process τ_s respectively.

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- Renewal function of the process H_s

$$h(x) = \int_0^\infty \mathbf{P}(H_s < x) ds$$

and its derivative

$$h'(x) = \int_0^\infty \mathbf{P}(H_s \in dx) ds/dx.$$

Theorem [M.Kwaśnicki, JM, M.Ryznar, AoP 2013]
For every $x, t > 0$ we have

$$\mathbf{P}(\overline{X_t} < x) \leq \min(1, \frac{e}{e-1}\kappa(1/t, 0)h(x)).$$

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Moreover, under some (mild but technical) assumptions on regularity of $\kappa(z, 0)$ we show existence of $C > 0$ such that

$$C \min(1, \kappa(1/t, 0)h(x)) \leq \mathbf{P}(\overline{X_t} < x).$$

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$$C \min(1, \kappa(1/t, 0)h(x)) \leq \mathbf{P}(\overline{X_t} < x).$$

- In particular, if $\kappa(z, 0)$ is regularly varying at 0 and ∞ then

$$\mathbf{P}(\overline{X_t} < x) \approx \min(1, \kappa(1/t, 0)h(x)), \quad x > 0, t > 0,$$

(possibly with different indices).

If we assume that X is symmetric, then $\kappa(z, 0) = \sqrt{z}$ and we have

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If we assume that X is symmetric, then $\kappa(z,0)=\sqrt{z}$ and we have

$$\mathbf{P}(\overline{X_t}<x)\approx \min\left(1,\frac{h(x)}{\sqrt{t}}\right),\quad x>0, t>0.$$

Theorem [M.Kwaśnicki, JM, M.Ryznar; AoP 2013]
Let $\Psi(\xi)$ be the Lévy-Khintchin exponent of a symmetric Lévy process X_t, which is not a compound Poisson process, and suppose that both $\Psi(\xi)$ and $\xi^2/\Psi(\xi)$ are increasing in $\xi>0$. Then

$$\frac{2}{5}\frac{1}{\sqrt{\Psi(1/x)}}\le h(x)\le 5\,\frac{1}{\sqrt{\Psi(1/x)}},\quad x>0.$$

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- In particular, under above-given assumptions we have

$$\mathbf{P}(\overline{X_t}<x)\approx \min\left(1,\frac{1}{\sqrt{t\Psi(1/x)}}\right), \quad x,t>0$$

and the constants appearing in the estimates are $1/20000$ and 10.

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Here we work assuming

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- (H1) is equivalent to apparently stronger condition that $p_t \in \mathcal{C}_0(\mathbf{R})$ for every $t > 0$.
- These conditions imply existence of $f_t(x)$ on $(0, \infty)$

What we should expect?

Since, in very general setting, we have

$$\int_0^x f_t(y)dy = \mathbf{P}(\overline{X_t} < x) \approx \min(1, h(x)\kappa(1/t,0)),$$

we can expect that

- $$f_t(x) \sim h'(x)\kappa(1/t,0),$$

  when $x \to 0^+$ or $t \to \infty$,
- and similarly

  $$f_t(x) \approx h'(x)\kappa(1/t,0),$$

  for t large and x small.

Asymptotic behaviour of $f_t(x)$, when $x \to 0^+$

Theorem [L.Chaumont, JM; AIHP (hopefully 2016)]
The density of the law of the past supremum of $(X, \mathbf{P})$ fulfils the following asymptotic behaviour,

$$\lim_{x \to 0^+} \frac{f_t(x)}{h'(x)} = \pi(t, \infty)$$

uniformly on $[t_0, \infty)$ for every fixed $t_0 > 0$.

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- If $\kappa(z, 0)$ is regularly varying at 0, then

$$\kappa(1/t, 0) \sim \Gamma(1 - \rho)\pi(t, \infty)$$

  as $t \to \infty$.

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The proof is based on

- L. Chaumont representation of $f_t(x)$ in terms of entrance law of the reflected excursions (AoP 2013)
- G. Uribe result (slightly generalized) on asymptotics of the entrance law (Bernoulli 2014)
- the upper bounds $\mathbf{P}(\overline{X_t} < x) \leq \min(1, \frac{e}{e-1}\kappa(1/t, 0)h(x))$
- Under our assumption we have $h(x)/h'(x) \to 0$ as $x \to 0$.

Asymptotic behaviour of $f_t(x)$, when $t \to \infty$

Theorem [L.Chaumont, JM; AIHP (hopefully 2016)]
If we additionally assume that $\pi(t,\infty)$ is regularly varying at ∞ then

$$\lim_{t\to\infty} \frac{f_t(x)}{\pi(t,\infty)} = h'(x)$$

uniformly in x on every compact subset of $(0,\infty)$.

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Note that the following conditions are equivalent

- $\pi(t,\infty)$ is regularly varying at infinity,
- $z \to \kappa(z,0)$ is regularly varying at 0,
- $\lim_{t\to\infty} \mathbf{P}(X_t \geq 0) = \rho$.

Estimates of f_t

Theorem [L.Chaumont, JM; AIHP 2016]
For every fixed $x_0, t_0 > 0$ there exist constants $c_1, c_2 > 0$ such that

$$c_1\, \pi(t,\infty) \leq \frac{f_t(x)}{h'(x)} \leq c_2\, \frac{1}{t} \int_0^t \pi(s,\infty) ds, \quad x \leq x_0, t \geq t_0$$

and if additionally $\pi(t,\infty)$ is regularly varying at ∞ then

$$f_t(x) \overset{x_0,t_0}{\approx} \pi(t,\infty) h'(x) \approx \kappa(1/t, 0) h'(x)$$

for $x \leq x_0$ and $t \geq t_0$.

Continuity of f_t

Under **(H1)**, continuity of f_t is equivalent to the continuity of h' in the following sense:

Theorem [L.Chaumont, JM; AIHP 2016]
The following conditions are equivalent:

- f_t is continuous at $x_0 > 0$ for every $t > 0$,
- f_t is continuous at $x_0 > 0$ for some $t > 0$,
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- h' is continuous at $x_0 > 0$.

Note that

- in many (many) cases h' is continuous on $(0, \infty)$.
- However, sometimes h' is not continuous.

Explicit formulas

Theorem [Kwaśnicki, JM, Ryznar; AoP 2013]
Suppose that X_t is a symmetric Lévy process with Lévy-Khintchin exponent $\Psi(\xi)$. Suppose that $\Psi(\xi)$ is increasing in $\xi > 0$. Then

$$\frac{1}{\pi}\int_0^\infty \frac{\xi\Psi'(\lambda)}{(\lambda^2+\xi^2)\sqrt{\Psi(\lambda)}} \exp\left[\frac{1}{\pi}\int_0^\infty \frac{\xi \log \frac{\lambda^2-\zeta^2}{\Psi(\lambda)-\Psi(\zeta)}}{\xi^2+\zeta^2} d\zeta\right] e^{-t\Psi(\lambda)} d\lambda$$

is a Laplace transform of $\overline{X_t}$ (i.e. $\mathbf{E} e^{-\xi\overline{X_t}}$).

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Note: The Baxter-Donsker formula

$$\int_0^\infty e^{-zt}\mathbf{E}e^{-\xi\overline{X_t}}dt = \frac{1}{\sqrt{z}}\exp\left(-\frac{1}{\pi}\int_0^\infty \frac{\xi\log(z+\Psi(\eta))}{\xi^2+\eta^2}d\eta\right).$$

Theorem [M.Kwaśnicki, JM, M.Ryznar; SPA 2013]
Suppose that the Lévy-Khintchin exponent $\Psi(\xi)$ of a symmetric Lévy process X_t satisfied $\Psi(\xi) = \psi(\xi^2)$ for CBF $\psi(\xi)$. For every $t > 0$ we have

$$\mathbf{P}(\overline{X_t} < x) = \frac{2}{\pi} \int_0^\infty \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} F_\lambda(x) e^{-t\psi(\lambda^2)} d\lambda, \quad x > 0.$$

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- Here $F_\lambda(z)$ are generalized eigenfunctions on half-line (Kwaśnicki, Studia Math. 2011).
- We have to impose some additional (technical) assumptions on Ψ.
- There is a quite long list of examples.

Since

$$\mathbf{P}(\overline{X}_t < x) = \mathbf{P}(\tau_x > t),$$

where

$$\tau_x = \inf\{t \geq 0 : X_t \geq x\}$$

is the first passage time through a barrier at level $x \geq 0$.

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Differentiating (in t) the formula

$$\mathbf{P}(\tau_x > t) = \frac{2}{\pi} \int_0^\infty \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} F_\lambda(x) e^{-t\psi(\lambda^2)} d\lambda$$

we get

Theorem [M.Kwaśnicki, JM, M.Ryznar; SPA 2013]
If [some technical assumptions on $\psi(\xi)$], then

$$\frac{d^n}{dt^n}\mathbf{P}(\tau_x > t) = (-1)^n \frac{2}{\pi} \int_0^\infty \sqrt{\psi'(\lambda^2)}(\psi(\lambda^2))^{n-\frac{1}{2}} F_\lambda(x) e^{-t\psi(\lambda^2)} d\lambda$$

Theorems [M.Kwaśnicki, JM, M.Ryznar; SPA 2013]
Under [some regularity of $\psi(\xi)$], we have

$$\lim_{t\to\infty} t^{n+1/2}\frac{d^n}{dt^n}\mathbf{P}(\tau_x > t) = \frac{(-1)^n\Gamma(n+1/2)}{\pi}h(x).$$

and

$$\lim_{x\to 0^+} \sqrt{\psi(1/x^2)}\frac{d^n}{dt^n}\mathbf{P}(\tau_x > t) = \frac{(-1)^n\Gamma(n+1/2)}{\pi\Gamma(1+\alpha)}\frac{1}{t^{n+1/2}}$$

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Moreover

$$(-1)^n \frac{d^n}{dt^n} \mathbf{P}(\tau_x > t) \approx \frac{1}{t^{n+1/2} \sqrt{\psi(1/x^2)}},$$

when $t\psi(1/x^2)$ is large enough.

What happens if we replace

$$\overline{X}_t = \sup_{s \le t} X_s \quad \textbf{by} \;\; M_t = \sup_{s \le t} |X_s| \textbf{?}$$

Equivalently, we replace the first passage time τ_x by

$$\tau_{(-r,r)} = \inf\{t > 0 : |X_t| > r\},$$

since $\mathbf{P}(M_t \ge r) = \mathbf{P}(\tau_{(-r,r)} \le t)$.

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- In general, it is a disaster...
- We lose all the machinery of fluctuation theory...
- Can we still say something about $\mathbf{P}(M_t > x)$ or $\mathbf{P}(\tau_{(-r,r)} > t)$? At least for Brownian motion?

- Let us consider the Brownian motion $W = (W_t)_{t \geq 0}$ in $\mathbb{R}^d$.
- As previously $M_t = \sup_{s \leq t} |W_s|$ and

$$\tau_{B(0,r)} = \inf\{t > 0 : |W_t| \notin B(0,r)\}$$

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- We consider more general problem of describing

$$\mathbf{P}^x(W_t \in dy, \tau_{B(0,r)} > t), \quad x, y \in \mathbb{R}^d, \quad t > 0.$$

  by looking for sharp estimates of its density

$$p_{B(0,r)}(t,x,y) = \mathbf{P}^x(W_t \in dy, \tau_{B(0,r)} > t)/dy$$

  (transition probability density of BM killed upon leaving a ball)
- For $r = 1$ we will simply write $B = B(0,1)$ and consequently $p_B(t,x,y)$ and τ_B.

Theorem [Zhang; JDE 2002]
There exists constants c_1 and c_2 such that

$$p_B(t,x,y) \approx \left(\frac{(1-|x|)(1-|y|)}{t} \wedge 1 \right) \frac{1}{t^{d/2}} \exp\left(-c_i \frac{|x-y|^2}{2t} \right)$$

for $t < 1$, where c_1 and c_2 appear in the lower and upper bounds respectively.

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- The Zhang's result is more general (for bounded $C^{1,1}$ domains) and then the expressions $1-|x|$ are replaced by the distance to the boundary $\delta_D(x)$.
- The upper-bounds were provided by Davies in 1987
- The bounds for t large are simple and well-known.

Classical Dirichlet heat kernel of a ball

Theorem [JM, G.Serafin 2016]

We have

$$p_B(t,x,y) \stackrel{d}{\approx} h(t,x,y)\frac{1}{t^{d/2}}\exp\left(-\frac{|x-y|^2}{2t}\right)$$

for every $x, y \in B(0,1)$ and t small enough. Here $h(t,x,y)$ is equal to

$$\left(\frac{(1-|x|)(1-|y|)}{t}\wedge 1\right)+\left[\frac{(1-|x|)|x-y|^2}{t}\wedge 1\right]\left[\frac{(1-|y|)|x-y|^2}{t}\wedge 1\right]$$

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- The Zhang's result

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We have

$$p_B(t,x,y) \stackrel{d}{\approx} h(t,x,y)\frac{1}{t^{d/2}}\exp\left(-\frac{|x-y|^2}{2t}\right)$$

for every $x, y \in B(0,1)$ and t small enough. Here $h(t,x,y)$ is equal to

$$\left(\frac{(1-|x|)(1-|y|)}{t}\wedge 1\right) + \left[\frac{(1-|x|)|x-y|^2}{t}\wedge 1\right]\left[\frac{(1-|y|)|x-y|^2}{t}\wedge 1\right]$$

Corollary:

$$k_B(t,x,y) \approx \left[\frac{1-|x|}{t} + \frac{|x-z|^2}{t}\left[1\wedge\frac{(1-|x|^2)|x-z|^2}{t}\right]\right]\frac{1}{t^{d/2}}e^{-\frac{|x-z|^2}{2t}}$$

for every $z \in \partial B$ and $x \in B$, $t < 1$. Here

$$k_B(t,x,z) = \mathbf{P}^x[W_{\tau_B} \in dz, \tau_{B(0,1)} \in dt]/dzdt$$

Thank you very much.