Asymptotics in random balls model

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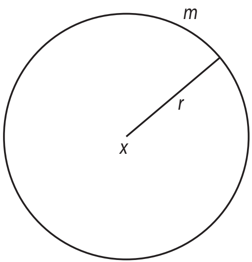
Talk based on

- B., C. Dombry. *Rescaled weighted random balls models and stable self-similar random fields*. SPA, 2009.
- B., C. Dombry. *Functional macroscopic behavior of weighted random ball model*. Alea, 2011.
- B., R. Gobard. *Infinite dimensional functional convergences in random balls model*. ESAIM, 2015.
- R. Gobard. *Random balls model with dependence*. J. Math. Anal. Appl., 2015.
- R. Gobard. *Random Balls generated by Ginibre Point Processes*. arXiv :1504.04513, 2015.
- R. Gobard. *Fluctuations dans les modèles de boules aléatoires*. Ph. D dissertation, Univ. Rennes 1, 2015.

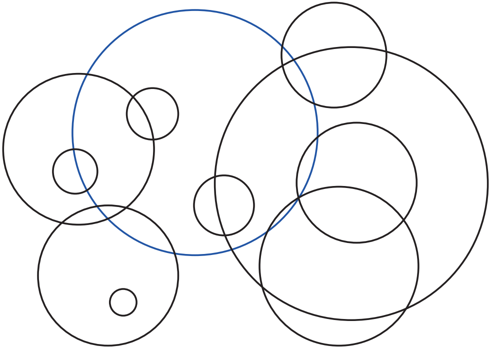
Euclidean ball $B(x, r)$



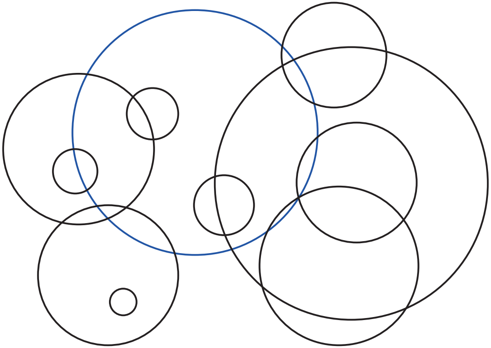
x = center
r = radius
m = weight

- $B(x, r) = x + rB(0, 1)$.

Balls model



Poissonian random balls model



(x_i, r_i, m_i) given by a (marked) Poisson point process N on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}$ with compensator

$$n(dx, dr, dm) = \lambda dx F(dr) G(dm)$$

Assumptions

- Radius :

$$\bar{F}(r) = F\big([r, +\infty[\big) \sim_{+\infty} C_\beta r^{-\beta}, \qquad d < \beta < \alpha d$$
$$\int_{\mathbb{R}_+} r^d F(dr) < +\infty$$

- Weight : G in the (normal) domain of attraction of a $S\alpha S$ with $1 < \alpha \leq 2$ (no weight $\rightsquigarrow \alpha = 2$).

[Crovella *et al.*, 98] : Heavy-tails in telecommunication network.

Modeling viewpoint

- $d = 1$: Traffic modeling : [Taqqu, Willinger, Sherman 97]
  $x =$ connection date
  $r =$duration of connection
  $m =$ intensity of connection
  whole picture (half balls)= traffic modeling

- $d = 2$: Telecommunication model : [Kaj 05]
  $x =$ antenna
  $r =$ range of emission
  $m =$ intensity of emission
  whole picture = covering network

- $d = 2$: Imaging (B&W picture) : [Biermé, Estrade 06]
  $x =$ pixel
  whole picture = grayscale in each pixel

- $d = 3$: Porous/granular media : [Biermé, Estrade 06]

Historical references

Typical behaviours : fractionnal, stable

- $d = 1$ (half-balls) : aggregation in teletraffic

[Cioczek-Georges and Mandelbrot, 95] : fBm

[Mikosch, Resnick,Rotzéen, Stegeman, 02] : fBm, stable process

[Kaj, Taqqu, 08] : Telecom process

- dim d (random fields)

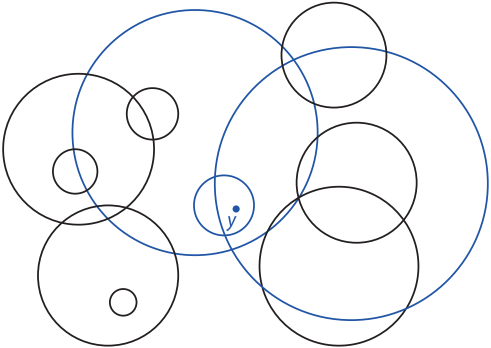
[Biermé, Estrade, 06] : Gaussian field, stable field, Takenaka field

[Kaj, Leskelä, Norros, Schmidt, 07] : Gaussian fields, stable fields

[Biermé, Demichel, Estrade, 09] : fractional Poisson fields

Shot noise contribution of the unweighted model

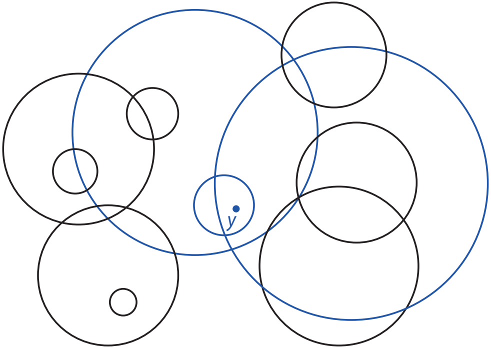
- Number of covering balls in each $y \in \mathbb{R}^d$



$$M(y) = \#\{(x_i, r_i) : y \in B(x_i, r_i)\}$$

Shot noise contribution of the unweighted model

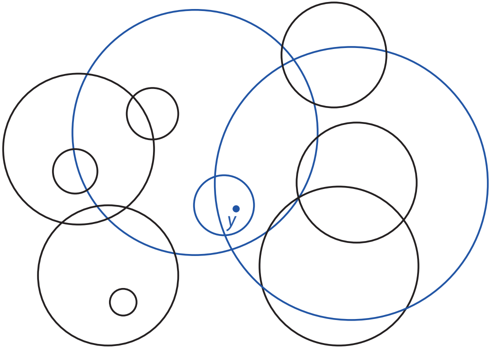
- Number of covering balls in each $y \in \mathbb{R}^d$



$$M(y) = \sum_i \mathbf{1}_{B(x_i,r_i)}(y) = \sum_i \delta_y\big(B(x_i,r_i)\big)$$
$$= \int \delta_y(B(x,r))N(dx,dr)$$

Shot noise contribution of the model

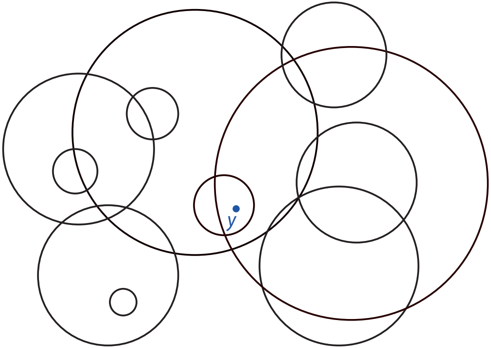
- Contribution of the weighted model in each $y \in \mathbb{R}^d$



$$
\begin{aligned}
M(y) &= \sum_i m_i \mathbf{1}_{B(x_i,r_i)}(y) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}} m\delta_y(B(x,r))N(dx,dr,dm).
\end{aligned}
$$

Shot noise contribution of the model

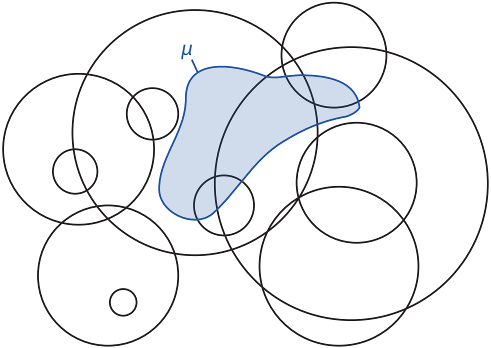
- Contribution of the weighted model in each $y \in \mathbb{R}^d$



$$M(\delta_y) = \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}} m\delta_y(B(x,r))N(dx,dr,dm).$$

Shot noise contribution of the model

- Contribution of the weighted model in a configuration $\mu \in \mathcal{M}$



$$M(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}} \mu(B(x,r))N(dx,dr,dm)$$

$\mathcal{M}$ = set of (signed) measures μ with finite total variation.

Mean contribution

$$
\begin{aligned}
\mathbb{E}\left[M(\mu)\right] &= \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}} m\mu((B(x,r))\ \lambda dx F(dr) G(dm) \\
&= \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}} m \int_{\mathbb{R}^d} \mathbf{1}_{B(x,r)}(y)\mu(dy)\ \lambda dx F(dr) G(dm) \\
&= \lambda \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}} m \int_{\mathbb{R}^d} \mathbf{1}_{B(y,r)}(x) dx \mu(dy)\ F(dr) G(dm) \\
&= \lambda \int_{\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}} m c_d r^d F(dr) G(dm) \mu(dy) \\
&= \lambda \left(c_d \int_{\mathbb{R}_+} r^d F(dr) \right) \left(\int_{\mathbb{R}} m G(dm) \right) \mu(\mathbb{R}^d).
\end{aligned}
$$

Scaling limit

- Apply a scaling $r \mapsto \rho r$ ($\rho \in \mathbb{R}_+$) : $B(x, r) \mapsto B(x, \rho r)$
- Two kinds of scaling : zoom-in ($\rho \to +\infty$) or zoom-out ($\rho \to 0$).
- The scaling limit
  gives insight in the local behaviour
  microscopic analysis

In the sequel, we focus on zoom-out asymptotics.

Scaling limit

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Scaling limit

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- Two kinds of scaling : zoom-in ($\rho \to +\infty$) or zoom-out ($\rho \to 0$).
- The scaling limit
  offers a distant view of the model
  provides a summary of the dependence structure

In the sequel, we focus on zoom-out asymptotics.

Macroscopic analysis

- Adapt the intensity of the PPP to the scaling :

$$\lambda(\rho) \to +\infty, \qquad \rho \to 0.$$

-

$$M_\rho(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m\,\mu(B(x,r))\,N_\rho(dx,dr,dm)$$

where $N_\rho(dx,dr,dm) =$ PPP with compensator

$$\lambda(\rho)dx F_\rho(dr)G(dm)$$

with $\bar{F}_\rho(r) = \bar{F}(r/\rho)$.

- Fluctuations :

$$\lim_{\rho\to 0} \frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n(\rho)}.$$

Configuration space $\mathcal{M}_{\alpha,\beta}$

Definition

$\mu \in \mathcal{M}_{\alpha,\beta}$ *if there exist* $C < +\infty$ *and* $0 < p < \beta < q$ *such that :*

$$\int_{\mathbb{R}^d} \left|\mu(B(x,r))\right|^{\alpha} dx \leq C \min(r^p, r^q).$$

▶ Properties : Linear space, stable by translations, rotations, dilatations, inclusion properties.

▶ Examples.

- $\mu = Leb_B$ uniform on bounded set B ($\rightsquigarrow$ cumulative regime).
- Conditions on moment.
- $\alpha = 2$: Riesz energy.

Heuristics of the regimes

- Number of large balls :

$$\begin{aligned}
\#\{i : 0 \in B(x_i, r_i), r_i \geq 1\} &= \sum_i \mathbf{1}_{B(x_i,r_i)}(0)\mathbf{1}_{\{r_i \geq 1\}} \\
&= \int_{\mathbb{R}^d \times \mathbb{R}_+} \mathbf{1}_{\{\|x\|<r\}}\mathbf{1}_{\{r>1\}}\lambda(\rho)dxF_\rho(dr) \\
&= c_d\lambda(\rho)\int_1^{+\infty} r^d F_\rho(dr) \\
&\sim_{\rho\to 0} \frac{c_d C_\beta}{\beta - d}\lambda(\rho)\rho^\beta.
\end{aligned}$$

- 3 regimes :
  - « Large balls » normalization : $\lambda(\rho)\rho^\beta \to +\infty$
  - « Intermediate balls » normalization : $\lambda(\rho)\rho^\beta \to \text{cte} \in (0, +\infty)$
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Large balls regime

Theorem

Hyp. $\lambda(\rho)\rho^\beta \to +\infty$.

Let $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\alpha}$*. We have :*

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n(\rho)} \overset{\mathcal{M}_{\alpha,\beta}}{\Longrightarrow} Z_\alpha(\cdot), \quad \rho \to 0$$

where

$$Z_\alpha(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x,r))\, M_\alpha(dr, dx)$$

and M_α *= stable measure controled by* $\sigma^\alpha C_\beta r^{-1-\beta} drdx$.

Heuristics when $m = 1$ ($\alpha = 2$)

- The limit is driven by large balls

$\Longrightarrow$ Asymptotic dependence

covariation

- $\lambda(\rho)\rho^\beta \to +\infty$: $\lambda(\rho) \to +\infty$ faster than $\rho^\beta \to 0$.

1. $\lambda \to +\infty$: superposition of infinite number of iid balls + CLT $\Rightarrow$ Gaussian limit.
2. $\rho \to 0$ shapes the covariance and the setting remains Gaussian.
3. When $m \neq 1$, $\alpha \in (1, 2]$: CLT $\leadsto$ stable domain of attraction.

$$\lim_{\lambda \to +\infty} \frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^\beta)^{1/2}} \overset{fdd}{\Longrightarrow} \text{Gaussian setting.}$$

details

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$$\lim_{\rho\to 0}\lim_{\lambda\to+\infty}\frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^\beta)^{1/2}} \overset{fdd}{\Longrightarrow} \text{Shaped Gaussian setting.}$$

details

Heuristics

- The limit is driven by large balls

$\Longrightarrow$ Asymptotic dependence

covariation

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$$\lim_{\rho\to 0}\lim_{\lambda\to+\infty} \frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^\beta)^{1/\alpha}} \overset{fdd}{\Longrightarrow} \alpha\text{-stable setting.}$$

details

Small balls regime

Theorem

Hyp. $d < \beta < \alpha d$, $\lambda(\rho)\rho^\beta \to 0$.

With $n(\rho) := (\lambda(\rho)\rho^\beta)^{1/\gamma}$ *with* $\gamma = \beta/d \in (1, \alpha)$, *we have :*

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n(\rho)} \xrightarrow{L^1(\mathbb{R}^d)\cap L^\alpha(\mathbb{R}^d)} \widetilde{Z}_\gamma(\cdot)$$

where for $\mu(dx) = \phi(x)dx$, $\widetilde{Z}_\gamma(\mu) = \int_{\mathbb{R}^d} \phi(x)\widetilde{M}_\gamma(dx)$, *is a* γ*-stable integral (with explicit parameters).*

Heuristics

- $\lambda(\rho)\rho^{\beta} \to 0$: $\rho^{\beta} \to 0$ faster than $\lambda(\rho) \to +\infty$.

1. $\rho \to 0$: the scaling kills the overlapping balls and yields independence.

   covariation

2. $\lambda \to +\infty$: F is heavy tailed $\Rightarrow$ the contributions of the non-overlapping balls are in a stable domain of attraction.

   $\mu(B(x,r)) \rightsquigarrow cr^{d} \rightsquigarrow (\beta/d)$-regular tail $\rightsquigarrow$ (β/d)-stable.

$$\lim_{\lambda\to+\infty} \lim_{\rho\to 0} \frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^{\beta})^{1/\gamma}} \xrightarrow{fdd} \gamma\text{-stable setting with independence.}$$

details

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details

Intermediate balls regime

Theorem

Hyp. $\lambda(\rho)\rho^\beta \to a^{d-\beta}$ *for* $a \in (0, +\infty)$.

We have

$$M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)] \overset{\mathcal{M}_{\alpha,\beta}}{\Longrightarrow} J(\mu_a)$$

where J *is the compensated Poisson integral*

$$J(\mu) = \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+} m\mu(B(x,r))\, \widetilde{N}_\beta(dx, dr, dm)$$

with N_β *= Poisson measure compensated by* $C_\beta r^{-\beta-1} dx dr G(dm)$.

Notation. dilatation of μ : $\mu_a(A) = \mu(a^{-1}A)$.

Heuristics. Take the limit in the compensator of the Poisson measure.

Properties of the limits $Z = \{Z_\alpha, \widetilde{Z}_\gamma, J\}$

- **Isotropy** : $\forall \Theta \in \mathcal{O}(\mathbb{R}^d)$, $Z(\Theta\mu) \stackrel{fdd}{=} Z(\mu)$.
- **Stationarity** : $\forall y \in \mathbb{R}^d$, $Z(\mu(\cdot - y)) \stackrel{fdd}{=} Z(\mu(\cdot))$.
- **Self-similarity** : $\forall a > 0$, let $\mu_a(A) = \mu(a^{-1}A)$

$$\begin{aligned} Z_\alpha(\mu_a) &\stackrel{fdd}{=} a^{(d-\beta)/\alpha} Z_\alpha(\mu) \\ \widetilde{Z}_\gamma(\mu_a) &\stackrel{fdd}{=} a^{(d-\beta)/\gamma} \widetilde{Z}_\gamma(\mu). \end{aligned}$$

Aggregative similarity

A (centered) random field $X = \{X(\mu) : \mu \in \mathcal{S} \subset \mathcal{M}\}$ is aggregate similar with index ρ if for $X^{(i)}$, $1 \leq i \leq m$, iid$\sim X$ we have

$$\sum_{i=1}^{m} X^{(i)}(\mu) \overset{fdd}{=} X(\mu_{m^\rho}).$$

- Ex. of aggregative similarity in dim 1, [Kaj 05] :

$$\sum_{i=1}^{m} X^{(i)}(t) \overset{fdd}{=} m^\rho X\left(m^{-\rho} t\right).$$

  - H-fBm : $\rho = \frac{1}{2(1-H)}$.
  - α-stable Lévy process : $\rho = \frac{1}{(\alpha-1)}$.
  - Poisson process : $\rho = 1$.

Aggregative similarity in balls model

- Aggregative similarity for J with $\rho = 1/(d-\beta)$:

$$J(\mu_{m^{1/(d-\beta)}}) \overset{fdd}{=} \sum_{i=1}^{m} J^i(\mu).$$

- Interpretation of $\lim_{\rho\to 0} \lambda(\rho)\rho^\beta = a^{d-\beta} = m$.

Each $(1/m)$-fraction of the balls generates one copy of J.

- When $a^{d-\beta} \to +\infty$,

$$\frac{1}{a^{(d-\beta)/\alpha}} J(\mu_a) \Longrightarrow Z_\alpha(\mu).$$

$\alpha = 2$ (no weight) : CLT

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- Interpretation of $\lim_{\rho\to 0} \lambda(\rho)\rho^{\beta} = a^{d-\beta} = m$.
Each $(1/m)$-fraction of the balls generates one copy of J.
- When $a^{d-\beta} = m \to +\infty$,

$$\frac{1}{a^{(d-\beta)/2}} \sum_{i=1}^{m} J^{(i)}(\mu) \Longrightarrow Z_2(\mu).$$

$\alpha = 2$ (no weight) : CLT

Aggregative similarity in balls model

- Aggregative similarity for J with $\rho = 1/(d-\beta)$:

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Each $(1/m)$-fraction of the balls generates one copy of J.
- When $a^{d-\beta} = m \to +\infty$,

$$\frac{1}{a^{(d-\beta)/\alpha}} J(\mu_a) \stackrel{\mathcal{L}}{=} \frac{1}{a^{(d-\beta)/\alpha}} \sum_{i=1}^{m} J^{(i)}(\mu) \Longrightarrow Z_\alpha(\mu).$$

$\alpha \in (1,2]$: stable domain of attraction

Aggregative similarity in balls model

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$$\frac{1}{a^{(d-\beta)/\alpha}} J(\mu_a) \Longrightarrow Z_\alpha(\mu).$$

Iterated limits

- $\lambda(\rho)\rho^{\beta} \to a^{d-\beta}$

$$\frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^{\beta})^{1/\alpha}} \Longrightarrow \frac{J(\mu_a)}{(a^{d-\beta})^{1/\alpha}} \Longrightarrow Z_\alpha(\mu)$$

- $\lambda(\rho)\rho^{\beta} \to a^{d-\beta}$

$$\frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^{\beta})^{d/\beta}} \Longrightarrow \frac{J(\mu_a)}{(a^{d-\beta})^{d/\beta}} \Longrightarrow \widetilde{Z}_\gamma(\mu)$$

- Poissonian bridge between stable fields ($a \to +\infty/0$) :

$$J(\mu_a)$$

$$\swarrow \qquad \searrow$$

$$\widetilde{Z}_\gamma(\mu) \qquad\qquad Z_\alpha(\mu)$$

[Kaj, Gaigalas 03], [Gaigalas 06]

Iterated limits

- $\lambda(\rho)\rho^{\beta} \to a^{d-\beta} \to +\infty$

$$\frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^{\beta})^{1/\alpha}} \Longrightarrow \frac{J(\mu_a)}{(a^{d-\beta})^{1/\alpha}} \Longrightarrow Z_\alpha(\mu)$$

- $\lambda(\rho)\rho^{\beta} \to a^{d-\beta}$

$$\frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^{\beta})^{d/\beta}} \Longrightarrow \frac{J(\mu_a)}{(a^{d-\beta})^{d/\beta}} \Longrightarrow \widetilde{Z}_\gamma(\mu)$$

- Poissonian bridge between stable fields ($a \to +\infty/0$) :

$$J(\mu_a)$$

$$\swarrow \qquad \searrow$$

$$\widetilde{Z}_\gamma(\mu) \qquad\qquad Z_\alpha(\mu)$$

[Kaj, Gaigalas 03], [Gaigalas 06]

Iterated limits

- $\lambda(\rho)\rho^{\beta} \to a^{d-\beta} \to +\infty$

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$$\begin{array}{ccc} & J(\mu_a) & \\ \swarrow & & \searrow \\ \widetilde{Z}_{\gamma}(\mu) & & Z_{\alpha}(\mu) \end{array}$$

[Kaj, Gaigalas 03], [Gaigalas 06]

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- $\lambda(\rho)\rho^{\beta} \to a^{d-\beta} \to 0$

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- Poissonian bridge between stable fields ($a \to +\infty/0$) :

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[Kaj, Gaigalas 03], [Gaigalas 06]

Poissonian bridge

- $\lambda(\rho)\rho^\beta \to a^{d-\beta} \to +\infty$

$$\frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^\beta)^{1/\alpha}} \Longrightarrow \frac{J(\mu_a)}{(a^{d-\beta})^{1/\alpha}} \Longrightarrow Z_\alpha(\mu)$$

- $\lambda(\rho)\rho^\beta \to a^{d-\beta} \to 0$

$$\frac{M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]}{(\lambda(\rho)\rho^\beta)^{d/\beta}} \Longrightarrow \frac{J(\mu_a)}{(a^{d-\beta})^{d/\beta}} \Longrightarrow \widetilde{Z}_\gamma(\mu)$$

- Poissonian bridge between stable fields ($a \to +\infty/0$) :

$$\begin{array}{ccc} & J(\mu_a) & \\ \swarrow & & \searrow \\ \widetilde{Z}_\gamma(\mu) & & Z_\alpha(\mu) \end{array}$$

[Kaj, Gaigalas 03], [Gaigalas 06]

Functional convergences

Functional convergences on some configuration set (compact, dominated)

- Large + intermediate balls regime : functional convergences
- Small balls regime : no hope

Ex. Uniform configurations on bounded set $\rightsquigarrow$ cumulative regime

Proof. Functional convergence $\rightsquigarrow$ tightness $\rightsquigarrow$ moment bounds

$$\mathbb{E}\left[|X|^\gamma\right] = A(\gamma) \int_0^{+\infty} \left(1 - |\varphi_X(\theta)|^2\right) \theta^{-1-\gamma} \, d\theta$$

Grain/fading model

- Grain model (anisotropic model)

$$\begin{aligned} B(0,1) &\rightsquigarrow C \\ B(x,r) &\rightsquigarrow x + rC \end{aligned}$$

$$M(\mu) = \int m\mu(x + rC) N(dx, dr, dm)$$

$\rightsquigarrow$ Similar type results

- Fading function (radially decreasing)

$$B(0,1) \text{ or } \mathbf{1}_{B(0,1)} \rightsquigarrow h.$$

$$M(\mu) = \int m \int h\Big(\frac{y-x}{r}\Big)\mu(dy) N(dx, dr, dm)$$

$\rightsquigarrow$ Similar type results

Stable asymptotics with dependence/independence prevail
Functional convergences prevail

Center-dependent radius balls model

Consider N a PPP with compensator

$$f(x,r)\,dxdrG(dm)$$

with

$$r \mapsto \|f(\cdot,r)\|_\infty \text{ continuous}$$
$$\int_{\mathbb{R}_+} r^d \|f(\cdot,r)\|_\infty dr < +\infty$$

and

$$f(x,r) \sim_{r\to+\infty} \frac{g(x)}{r^{\beta(x)+1}},$$

- $g(x) \rightsquigarrow$ inhomogeneity
- $\beta(x) \rightsquigarrow$ center-dependent radius

$$d < \beta_1 \leq \beta(x) \leq \beta_2 < \alpha d$$

Rescaled center-dependent radius balls model

Consider N_ρ a PPP with compensator

$$f_\rho(x,r)\ dxdrG(dm)$$

with

$$r \mapsto \|f_\rho(\cdot,r)\|_\infty \text{ continuous}$$
$$\int_{\mathbb{R}_+} r^d \|f_\rho(\cdot,r)\|_\infty dr < +\infty$$

and

$$f_\rho(x,r) \sim_{r\to+\infty} \frac{g_\rho(x)}{r^{\beta(x)+1}}, \quad g_\rho(x) = \lambda(\rho)g(x)$$

- Fluctuations of

$$M_\rho(\mu) = \int m\mu\big(B(x,r)\big)\ N_\rho(dx,dr,dm)$$

Heuristics

- Zoom-out scaling $\Longrightarrow$ the most large ball prevail (radius index β_1)
- Hyp. :

$$Leb\big(x \in \mathbb{R}^d : \beta(x) = \beta_1\big) > 0$$

- Regimes driven by

$$\lambda(\rho)\rho^{\beta_1}$$

Stable regime with dependence

Theorem

Hyp. $\lambda(\rho)\rho^{\beta_1} \to +\infty$.

Let $n(\rho) = (\lambda(\rho)\rho^{\beta_1})^{1/\alpha}$. *We have :*

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n(\rho)} \overset{\mathcal{M}_{\alpha,\beta_1,\beta_2}}{\Longrightarrow} Z_\alpha(\cdot), \quad \rho \to 0$$

where

$$Z_\alpha(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x,r))\, M_\alpha(dr, dx)$$

and M_α *= stable measure controled by* $C_\beta \sigma^\alpha g(x) \mathbf{1}_{B_1}(x) r^{-1-\beta_1} drdx$.

Stable regime with independence

Theorem

Hyp. $d < \beta_1 \leq \beta_2 < \alpha d$, $\lambda(\rho)\rho^{\beta_1} \to 0$.

With $n(\rho) := (\lambda(\rho)\rho^{\beta_1})^{1/\gamma}$ *with* $\gamma = \beta_1/d \in (1, \alpha)$, *we have :*

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n(\rho)} \xrightarrow{L^1(\mathbb{R}^d)\cap L^\alpha(\mathbb{R}^d)} \widetilde{Z}_\gamma(\cdot)$$

where for $\mu(dx) = \phi(x)dx$, $\widetilde{Z}_\gamma(\mu) = \int_{\mathbb{R}^d} \phi(x)\widetilde{M}_\gamma(dx)$, *is a* γ*-stable integral with control measure* $\sigma^\gamma g(x)\mathbf{1}_{B_1}(x)dx$.

Poissonian bridge regime

Theorem

Hyp. $\lambda(\rho)\rho^{\beta_1} \to a^{d-\beta}$ *for* $a \in (0, +\infty)$.

We have

$$M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)] \overset{\mathcal{M}_{\alpha,\beta_1,\beta_2}}{\Longrightarrow} J(\mu_a),$$

where J *is the compensated Poisson integral*

$$J(\mu) = \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+} m\mu(B(x,r)) \, \widetilde{N}_{\beta_1}(dx, dr, dm)$$

with N_β *= Poisson measure compensated by*

$$C_\beta r^{-\beta_1 - 1} g(x) \mathbf{1}_{B_1}(x) dxdrG(dm).$$

Comments

- Similar properties for $Z_\alpha, \widetilde{Z}\gamma, J$.
- Similar generalizations for gain model/fading model.
- SImilarly, zoom-in scaling relies on β_2.

Ginibre balls model in $\mathbb{R}^2 \sim \mathbb{C}$

Unweighted ($\alpha = 2$) random balls given by a marked Ginibre point process

- Centers = Ginibre point process with kernel

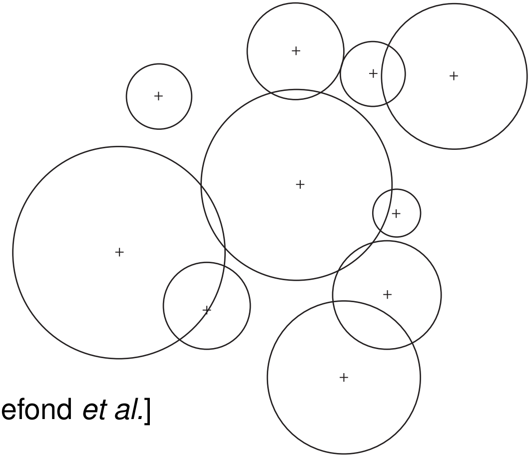
$$K(x,y) = \frac{\lambda}{\pi} \exp\Big(-\frac{\lambda}{2}(|x|^2 + |y|^2)\Big) e^{\lambda x \bar{y}}$$

Joint intensities

$$\rho_n(x_1, \dots, x_n) = \det\big(K(x_i, x_j)\big)$$

Ginibre balls model in $\mathbb{R}^2 \sim \mathbb{C}$

- Ginibre PP exhibits repulsiveness



- [Decreusefond *et al.*]

Ginibre balls model in $\mathbb{R}^2 \sim \mathbb{C}$

- Radii iid $\sim F(dr) = f(r)dr$ with

$$f(r) \sim \frac{C_\beta}{r^{1+\beta}}, \quad 2 < \beta < 4.$$

- Centers+radii = determinantal point process N with kernel

$$\widehat{K}((x,r),(y,s)) = \sqrt{f(r)}K(x,y)\sqrt{f(s)}.$$

- Contribution on the model in μ :

$$M(\mu) = \int \mu(B(x,r))N(dx,dr).$$

Mean contribution

- $$
\begin{aligned}
\mathbb{E}\left[M(\mu)\right] &= \int_{\mathbb{C}\times\mathbb{R}_+} \mu\big(B(x,r)\big)\ \widehat{K}\big((x,r),(x,r)\big)\,dxdr \\
&= \int_{\mathbb{C}\times\mathbb{R}_+} \mu\big(B(x,r)\big)\ \frac{\lambda}{\pi} f(r)\,dxdr \\
&= \frac{1}{\pi}\mu(\mathbb{C})\lambda \underbrace{\int \pi r^2 f(r)\,dr}_{\text{mean volume}}
\end{aligned}
$$

Rescaled Ginibre model

- N_ρ determinantal PP on $\mathbb{C} \times \mathbb{R}_+$ with kernel

$$\widehat{K}_\rho\big((x,r),(y,s)\big) = \sqrt{\frac{f(r/\rho)}{\rho}}\, \underbrace{\frac{\lambda(\rho)}{\pi} e^{-\frac{\lambda(\rho)}{2}(|x|^2+|y|^2)} e^{\lambda(\rho)x\bar{y}}}_{K_\rho(x,y)} \sqrt{\frac{f(s/\rho)}{\rho}}.$$

- $$M_\rho(\mu) = \int \mu\big(B(x,r)\big)\; N_\rho(dx,dr).$$

Mean number of large balls

$$\begin{aligned}
&\mathbb{E}\left[\#\{(x,r)\in N_\rho : 0\in B(x,r), r>1\}\right]\\
&= \int_{\{(x,r):0\in B(x,r),r>1\}} \widetilde{K}_\rho\big((x,r),(x,r)\big)\,dxdr\\
&= \int_{\{(x,r):0\in B(x,r),r>1\}} \frac{\lambda(\rho)}{\pi}\frac{f(r/\rho)}{\rho}\,dxdr\\
&= \lambda(\rho)\int_1^{+\infty} r^2\frac{f(r/\rho)}{\rho}\,dr\\
&\sim \lambda(\rho)\rho^\beta\int_1^{+\infty} r^{-1-\beta}\,dr.
\end{aligned}$$

$\rightsquigarrow$ 3 regimes :

$$\lambda(\rho)\rho^\beta \longrightarrow 0/a/+\infty.$$

Large balls regime

Theorem

Hyp. $\lambda(\rho)\rho^\beta \to +\infty$.

Let $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/2}$. *We have :*

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n(\rho)} \overset{\mathcal{M}_{2,\beta}}{\Longrightarrow} Z_2(\cdot), \quad \rho \to 0$$

where

$$Z_\alpha(\mu) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu(B(x,r))\, M_2(dr, dx)$$

and M_2 = *Gaussian random measure controled by* $\frac{C_\beta}{\pi} r^{-1-\beta} drdx$.

Small balls regime

Theorem

Hyp. $2 < \beta < 4$, $\lambda(\rho)\rho^{\beta} \to 0$.

With $n(\rho) := (\lambda(\rho)\rho^{\beta})^{1/\gamma}$ *with* $\gamma = \beta/2 \in (1,2)$, *we have :*

$$\frac{M_\rho(\cdot) - \mathbb{E}[M_\rho(\cdot)]}{n(\rho)} \xrightarrow{L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)} \widetilde{Z}_\gamma(\cdot)$$

where for $\mu(dx) = \phi(x)dx$, $\widetilde{Z}_\gamma(\mu) = \int_{\mathbb{R}^d} \phi(x)\widetilde{M}_\gamma(dx)$, *is a* γ*-stable integral (with explicit parameters)).*

Intermediate balls regime

Theorem

Hyp. $\lambda(\rho)\rho^\beta \to a^{d-\beta}$ *for* $a \in (0, +\infty)$.

We have

$$M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)] \overset{\mathcal{M}_{\alpha,\beta}}{\Longrightarrow} J(\mu_a), \quad \rho \to 0$$

where J *is the compensated Poisson integral*

$$J(\mu) = \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^+} \mu(B(x,r))\ \widetilde{N}_\beta(dx, dr)$$

with N_β *= Poisson measure compensated by* $\frac{C_\beta}{\pi} r^{-\beta-1} dxdr$.

Heuristics

- Laplace transform of $\widetilde{M}_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]$:

$$\mathbb{E}\left[\exp\left(\theta n(\rho)^{-1}\widetilde{M}_\rho(\mu)\right)\right]$$

$$= \exp\left(\sum_{n\geq 1}\frac{(-1)^n}{n}Tr\left(\widehat{K}_\rho\left[1-e^{-\theta n(\rho)\mu(B(\cdot,\cdot))}\right]^n\right)\right)$$

$$= \exp\left(-\int_{\mathbb{C}\times\mathbb{R}_+}\psi(\theta n(\rho)\mu(B(x,r)))\frac{\lambda(\rho)}{\pi\rho}f(r/\rho)dxdr\right)$$

$$\times\exp\left(\sum_{n\geq 2}\frac{(-1)^n}{n}Tr\left(\widehat{K}_\rho\left[1-e^{-\theta n(\rho)\mu(B(\cdot,\cdot))}\right]^n\right)\right)$$

with

$$\widehat{K}_\rho[f]g(x,r) = \int_{\mathbb{C}\times\mathbb{R}_+}\sqrt{f(x,r)}K_\rho(x,y)\sqrt{f(y,s)}\,g(y,s)\,dyds.$$

Heuristics

- Laplace transform of $\widetilde{M}_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]$:

$$\mathbb{E}\Big[\exp\big(\theta n(\rho)^{-1}\widetilde{M}_\rho(\mu)\big)\Big]$$

$$= \exp\left(\sum_{n\geq 1}\frac{(-1)^n}{n} Tr\Big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)\mu(B(\cdot,\cdot))}\big]^n\Big)\right)$$

$$= \exp\left(-\int_{\mathbb{C}\times\mathbb{R}_+}\psi(\theta n(\rho)\mu(B(x,r)))\frac{\lambda(\rho)}{\pi\rho}f(r/\rho)dxdr\right)$$

$$\times \exp\left(\sum_{n\geq 2}\frac{(-1)^n}{n} Tr\Big(\widehat{K}_\rho\big[1-e^{-\theta n(\rho)\mu(B(\cdot,\cdot))}\big]^n\Big)\right)$$

with

$$\psi(u) = e^{-u} - 1 + u.$$

Heuristics

- Term #1 : Laplace transform of a Poisson integral

$$n(\rho)^{-1}\int_{\mathbb{C}\times\mathbb{R}_+}\mu\big(B(x,r)\big)\Pi_\rho(dx,dr)$$

$\rightsquigarrow$ Similar traitment

- Other terms : geometric control

$$Tr\Big(\widetilde{K}_\rho\big[1-\exp\big(-\theta n(\rho)^{-1}\mu(B(\cdot,\cdot))\big)\big]^n\Big)$$
$$\leq \quad Tr\Big(\widetilde{K}_\rho\big[1-\exp\big(-\theta n(\rho)^{-1}\mu(B(\cdot,\cdot))\big)\big]^2\Big)^{n/2}$$

and

$$\lim_{\rho\to 0} Tr\Big(\widetilde{K}_\rho\big[1-\exp\big(-\theta n(\rho)^{-1}\mu(B(\cdot,\cdot))\big)\big]^n\Big)=0.$$

Poisson/Ginibre/Determinantal

- The $\mathcal{K}$-function measures the distribution of the inner points distance :

$$\text{(Scaled Ginibre) } \mathcal{K}_c(r) = \pi r^2 - \frac{\pi}{c}\left(1 - e^{-cr^2}\right) \xrightarrow{c \to +\infty} \pi r^2 \text{ (Poisson)}$$

- Weight ?
- Zoom-in ?
- General determinantal PP ?
- Robustness ?

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Technical heuristics

Characteristic function of $\widetilde{M}_\rho(\mu) = n(\rho)^{-1}\big(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]\big)$

$$
\begin{aligned}
&\varphi_{\widetilde{M}_\rho(\mu)}(\theta) \\
&= \exp\Big(\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}} \psi\big(n(\rho)^{-1}\theta m\mu(B(x,r))\big)\,\lambda(\rho)dxF_\rho(dr)G(dm)\Big) \\
&= \exp\Big(\int_{\mathbb{R}^d\times\mathbb{R}_+} \psi_G\big(n(\rho)^{-1}\theta\mu(B(x,r))\big)\,\lambda(\rho)dxF_\rho(dr)\Big) \\
&\sim \exp\Big(\int_{\mathbb{R}^d\times\mathbb{R}_+} -\sigma^\alpha n(\rho)^{-\alpha}|\theta|^\alpha|\mu(B(x,r))|^\alpha\lambda(\rho)dx\frac{\rho^\beta}{r^{1+\beta}}dr\Big) \\
&\to \varphi_{Z_\alpha}(\theta)
\end{aligned}
$$

with

$$\psi(u) = e^{iu} - 1 - iu.$$

Technical heuristics

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&= \exp\Big(\int_{\mathbb{R}^d\times\mathbb{R}_+} \psi_G\big(n(\rho)^{-1}\theta\mu(B(x,r))\big)\ \lambda(\rho)dxF_\rho(dr)\Big)\\
&\sim \exp\Big(\int_{\mathbb{R}^d\times\mathbb{R}_+} -\sigma^\alpha n(\rho)^{-\alpha}|\theta|^\alpha|\mu(B(x,r))|^\alpha\lambda(\rho)dx\frac{\rho^\beta}{r^{1+\beta}}dr\Big)\\
&\to \varphi_{Z_\alpha}(\theta)
\end{aligned}$$

with

$$\psi_G(u) = \int_{\mathbb{R}} \psi(mu)G(dm).$$

Technical heuristics

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&= \exp\Big(\int_{\mathbb{R}^d\times\mathbb{R}_+} \psi_G\big(n(\rho)^{-1}\theta\mu(B(x,r))\big)\,\lambda(\rho)dxF_\rho(dr)\Big) \\
&\sim \exp\Big(\int_{\mathbb{R}^d\times\mathbb{R}_+} -\sigma^\alpha n(\rho)^{-\alpha}|\theta|^\alpha|\mu(B(x,r))|^\alpha\lambda(\rho)dx\frac{\rho^\beta}{r^{1+\beta}}dr\Big) \\
&\to \varphi_{Z_\alpha}(\theta)
\end{aligned}
$$

with

$$\psi_G(u) \sim_{u\to 0} -\sigma^\alpha|u|^\alpha.$$

Technical heuristics

Characteristic function of $\widetilde{M}_\rho(\mu) = n(\rho)^{-1}\left(M_\rho(\mu) - \mathbb{E}\left[M_\rho(\mu)\right]\right)$

$$
\begin{aligned}
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&= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}_+} \psi_G\left(n(\rho)^{-1}\theta\mu(B(x,r))\right)\lambda(\rho)dxF_\rho(dr)\right) \\
&\sim \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}_+} -\sigma^\alpha n(\rho)^{-\alpha}|\theta|^\alpha|\mu(B(x,r))|^\alpha\lambda(\rho)dx\frac{\rho^\beta}{r^{1+\beta}}dr\right) \\
&\to \varphi_{Z_\alpha}(\theta)
\end{aligned}
$$

with

.

back

Covariation

- (X_1, X_2) α-stable vector

$$[X_1, X_2]_\alpha = \frac{1}{\alpha} \frac{\partial \sigma^\alpha(\theta_1, \theta_2)}{\partial \theta_1}_{|\theta_1=0, \theta_2=1}$$

where $\sigma(\theta_1, \theta_2)$ =scale parameter of $\theta_1 X_1 + \theta_2 X_2$.

Prop.

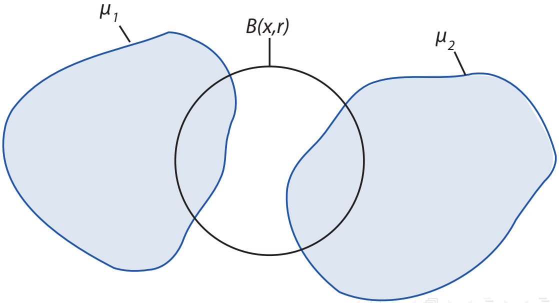
$$X_1 \perp\!\!\!\perp X_2 \quad \Longrightarrow \quad [X_1, X_2]_\alpha = 0.$$

Stable regime with dependence

- $\mathrm{Supp}(\mu_1) \cap \mathrm{Supp}(\mu_2) = \emptyset \not\Rightarrow Z_\alpha(\mu_1) \perp\!\!\!\perp Z_\alpha(\mu_2)$.

Covariation in our setting :

$$\begin{aligned}\left[Z_\alpha(\mu_1), Z_\alpha(\mu_2)\right]_\alpha &= \int_{\mathbb{R}^d \times \mathbb{R}^+} \mu_1(B(x,r))\mu_2(B(x,r))^{\langle \alpha-1 \rangle}\, \sigma^\alpha C_\beta r^{-1-\beta} dr dx \\ &\neq 0\end{aligned}$$



Technical heuristics

Characteristic function of $\widetilde{M}_\rho(\mu) = n(\rho)^{-1}\left(M_\rho(\mu) - \mathbb{E}\left[M_\rho(\mu)\right]\right)$

$$
\begin{aligned}
&\varphi_{\widetilde{M}_\rho(\mu)}(\theta) \\
&= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}} \psi\left(n(\rho)^{-1}\theta m \mu(B(x,r))\right)\lambda(\rho) dx F_\rho(dr) G(dm)\right) \\
&\sim \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}} \psi\left(n(\rho)^{-1}\theta m c_d \phi(x) r^d\right)\lambda(\rho) dx \frac{\rho^\beta}{r^{1+\beta}} dr G(dm)\right) \\
&= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}} \psi(s)\left(\frac{\theta m c_d \phi(x)}{n(\rho)}\right)^{\beta/d} \lambda(\rho)\rho^\beta \frac{s^{-1-\beta/d}}{d} ds dx G(dm)\right) \\
&\rightsquigarrow \varphi_{\beta/d}(\theta)
\end{aligned}
$$

with

$$
s = \frac{\theta m c_d \phi(x)}{n(\rho)} r^d.
$$

Technical heuristics

Characteristic function of $\widetilde{M}_\rho(\mu) = n(\rho)^{-1}\big(M_\rho(\mu) - \mathbb{E}\big[M_\rho(\mu)\big]\big)$

$$
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&\sim \exp\Big(\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}} \psi\big(n(\rho)^{-1}\theta m c_d\phi(x) r^d\big)\ )\ \lambda(\rho)dx\frac{\rho^\beta}{r^{1+\beta}}drG(dm)\Big) \\
&= \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}} \psi(s)\Big(\frac{\theta m c_d\phi(x)}{n(\rho)}\Big)^{\beta/d}\lambda(\rho)\rho^\beta\frac{s^{-1-\beta/d}}{d}\ dsdxG(dm)\right) \\
&\rightsquigarrow \varphi_{\beta/d}(\theta)
\end{aligned}
$$

with

$$
s = \frac{\theta m c_d \phi(x)}{n(\rho)} r^d.
$$

Stable regime with independence

- Heuristics : The limit is driven by small balls
At the limit, the small balls do not overlap
$\Longrightarrow$ Asymptotic independence.

- $$\mathrm{Supp}(\mu_1) \cap \mathrm{Supp}(\mu_2) = \emptyset \quad \Longrightarrow \quad \widetilde{Z}_\gamma(\mu_1) \perp\!\!\!\perp \widetilde{Z}_\gamma(\mu_2).$$

Covariation :

$$\begin{aligned} \left[\widetilde{Z}_\gamma(\mu_1), \widetilde{Z}_\gamma(\mu_2)\right]_\alpha &= \int_{\mathbb{R}^d} \phi_1(x)\phi_2(x)^{\langle\alpha-1\rangle}\widetilde{m}(dx) \\ &= 0 \end{aligned}$$

back

Configurations space

Definition

$\mathcal{M}_{\alpha,\beta_1,\beta_2}$ is the set of measure $\mu \in \mathcal{M}$ such that for $p < \beta_1 < \beta_2 < q$

$$\int \left|\mu(B(x,r)\right|^{\alpha} dx \leq C \min(r^p, r^q)$$

Proposition

- *$\mathcal{M}_{\alpha,\beta_1,\beta_2}$ is a linear space included in the space of diffuse measure when $\beta_1 > d$*
- *$\mathcal{M}_{\alpha,\beta_1,\beta_2}$ closed by rotation and dilatation*
- $\alpha \leq \alpha' \implies \mathcal{M}_{\alpha,\beta_1,\beta_2} \subset \mathcal{M}_{\alpha',\beta_1',\beta_2}$
- $\beta_1 \leq \beta_1' \leq \beta_2 \leq \beta_2' \implies \mathcal{M}_{\alpha,\beta_1,\beta_2} \subset \mathcal{M}_{\alpha,\beta_1',\beta_2'}$
- *$d < \beta_1 \leq \beta_2 \leq \alpha d$, $L^1(\mathbb{R}^d) \cap L^{\alpha}(\mathbb{R}^d) \subset \mathcal{M}_{\alpha,\beta_1,\beta_2}$.*