Spectrally one-sided Lévy processes with Parisian and classical reflection. [1]

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[1] joint work with F. Avram, and K. Yamazaki.

Spectrally ngative Lévy process.

- Let $X = (X(t); t \geq 0)$ be a spectrally positive Lévy process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Assume that its Laplace exponent $\psi : [0, \infty) \to \mathbb{R}$, i.e.

$$\mathbb{E}\left[\mathrm{e}^{-\theta X(t)}\right] =: \mathrm{e}^{\psi(\theta) t}, \qquad t, \theta \geq 0,$$

is given, by the *Lévy-Khintchine formula*

$$\psi(\theta) := \gamma\theta + \frac{\sigma^2}{2}\theta^2 + \int_{(0,\infty)} \left(\mathrm{e}^{-\theta z} - 1 + \theta x \mathbf{1}_{\{z<1\}}\right) \Pi(dz), \quad \theta \geq 0,$$

where $\gamma \in \mathbb{R}$, $\sigma \geq 0$, and Π is a measure on $(0, \infty)$ called the Lévy measure of X that satisfies

$$\int_{(0,\infty)} (1 \wedge z^2) \Pi(dz) < \infty.$$

- For the rest of the paper, we assume that

$$\mathbb{E}[X_1] = -\psi'(0+) < \infty.$$

Lévy processes with Parisian reflection below.

- Let $\mathcal{T}_r = \{T(i); i \geq 1\}$ be an increasing sequence of jump times of an independent Poisson process with rate $r > 0$. We construct the *Lévy process with Parisian reflection below* $X_r = (X_r(t); t \geq 0)$ as follows: the process is only observed at times $\mathcal{T}_r$ and is pushed up to 0 if only if it is below 0.
- More specifically, we have

$$X_r(t) = X(t), \quad 0 \leq t < T_0^-(1)$$

where

$$T_0^-(1) := \inf\{T(i) : \ X(T(i)) < 0\};$$

The process then jumps upward by $|X(T_0^-(1))|$ so that $X_r(T_0^-(1)) = 0$.
- For $T_0^-(1) \leq t < T_0^-(2) := \inf\{T(i) > T_0^-(1) : \ X_r(T(i)-) < 0\}$, we have $X_r(t) = X(t) + |X(T_0^-(1))|$.
- Suppose $R_r(t)$ is the cumulative amount of (Parisian) reflection until time $t \geq 0$. Then we have

$$X_r(t) = X(t) + R_r(t), \quad t \geq 0,$$

with

$$R_r(t) := \sum_{T_0^-(i) \leq t} |X_r(T_0^-(i)-)|, \quad t \geq 0,$$

Lévy processes with Parisian reflection below and classical reflection above.

- The process Y_r^b with additional (classical) reflection above can be defined analogously. Fix $b > 0$. Let

$$Y^b(t) := X(t) - L^b(t) \quad \text{where } L^b(t) := \sup_{0 \le s \le t} (X(s) - b) \vee 0, \quad t \ge 0,$$

be the process reflected from above at b.

- We have

$$Y_r^b(t) = Y^b(t), \quad 0 \le t < \widehat{T}_0^-(1)$$

where $\widehat{T}_0^-(1) := \inf\{T(i) : \; Y^b(T(i)) < 0\}$. The process then jumps upward by $|Y^b(\widehat{T}_0^-(1))|$ so that $Y_r^b(\widehat{T}_0^-(1)) = 0$.

- For $\widehat{T}_0^-(1) \le t < \widehat{T}_0^-(2) := \inf\{T(i) > \widehat{T}_0^-(1) : \; Y_r^b(T(i)-) < 0\}$, $Y_r^b(t)$ is the reflected process of $X(t) - X(\widehat{T}_0^-(1))$.
- It is clear that it admits a decomposition

$$Y_r^b(t) = X(t) + R_r^b(t) - L_r^b(t), \quad t \ge 0,$$

where $R_r^b(t)$ and $L_r^b(t)$ are, respectively, the cumulative amounts of Parisian and classical reflection until time t.

Scale functions

- Fix $q \geq 0$. Let $W^{(q)}$ be the scale function of X. Namely, this is a mapping from $\mathbb{R}$ to $[0, \infty)$ that takes the value zero on the negative half-line, while on the positive half-line it is a strictly increasing function that is defined by its Laplace transform:

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$

where

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\}.$$

- We also define, for $x \in \mathbb{R}$,

$$Z^{(q)}(x) := 1 + q \int_0^x W^{(q)}(y)dy, \qquad \text{and,} \qquad \overline{Z}^{(q)}(x) := \int_0^x Z^{(q)}(z)dz.$$

- Also, for $a < 0$ and $x \in \mathbb{R}$,

$$W_a^{(q,r)}(x) := \mathcal{M}_a^{(q,r)} W^{(q)}(x), \quad Z_a^{(q,r)}(x) := \mathcal{M}_a^{(q,r)} Z^{(q)}(x),$$
$$\overline{Z}_a^{(q,r)}(x) := \mathcal{M}_a^{(q,r)} \overline{Z}^{(q)}(x),$$

where we define, for any function $f : \mathbb{R} \to \mathbb{R}$,

$$\mathcal{M}_a^{(q,r)} f(x) := f(x-a) + r \int_0^x W^{(q+r)}(x-y) f(y-a)dy, \quad x \in \mathbb{R}, \quad a \in \mathbb{R}.$$

Identities.

- We are interested in computing the following quantities:
- For fixed $r > 0$ and define

$$\tau_a^-(r) := \inf\{t > 0 : X_r(t) < a\} \quad \text{and} \quad \tau_a^+(r) := \inf\{t > 0 : X_r(t) > a\}, \quad a < 0.$$

- Also

$$\eta_a^-(r) := \inf\{t > 0 : Y_r^b(t) < a\}, \quad a < 0,$$

- Identities for the process X_r:

$$\mathbb{E}_x\left(e^{-q\tau_b^+(r)-\theta R_r(\tau_b^+(r))}; \tau_b^+(r) < \tau_a^-(r)\right), \quad \mathbb{E}_x\left(e^{-q\tau_a^-(r)-\theta R_r(\tau_a^-(r))}; \tau_a^-(r) < \tau_b^+(r)\right)$$

$$\mathbb{E}_x\left(\int_0^{\tau_b^+(r)\wedge\tau_a^-(r)} e^{-qt} dR_r(t)\right).$$

- Identities for the process Y_r^b:

$$\mathbb{E}_x\left(e^{-q\eta_a^-(r)-\theta R_r^b(\eta_a^-(r))}; \eta_a^-(r) < \infty\right), \quad \mathbb{E}_x\left(\int_0^{\eta_a^-(r)} e^{-qt} dL_r^b(t)\right)$$

$$\mathbb{E}_x\left(\int_0^{\eta_a^-(r)} e^{-qt} dR_r^b(t)\right).$$

Two key results.

As in Loeffen etal. [6], for any $p \geq 0$, let $\mathcal{V}_0^{(p)}$ be the set of measurable functions $v_p : \mathbb{R} \to [0, \infty)$ satisfying

$$\mathbb{E}_x\left(e^{-p\tau_0^-} v_p(X(\tau_0^-)); \tau_0^- < \tau_b^+\right) = v_p(x) - \frac{W_p(x)}{W_p(b)} v_p(b), \quad x \leq b.$$

We shall further define $\tilde{\mathcal{V}}_0^{(p)}$ to be the set of positive measurable functions $v_p(x)$ that satisfy:

(i) For the case X is of bounded variation, $v_p \in \mathcal{V}_0^{(p)}$ and there exists large enough λ such that $\int_0^\infty e^{-\lambda z} v_p(z) dz < \infty$.

(ii) For the case X is of unbounded variation, there exist a sequence of functions $v_{p,n}$ that converge to v_p uniformly on compact sets, where $v_{p,n}$ belongs to the class $\tilde{\mathcal{V}}_0^{(p)}$ for the process X^n; here $(X^n; n \geq 1)$ is a sequence of spectrally negative Lévy processes of bounded variation that converge to X almost surely uniformly on compact time intervals.

Two key results.

Theorem (Simplifying formula for the excursion measure away from 0.)

Consider functions $w_p, v_q : \mathbb{R} \to [0,\infty)$ that satisfy the following:

1. w_p *and* v_q *belong to the classes* $\tilde{\mathcal{V}}_0^{(p)}$ *and* $\tilde{\mathcal{V}}_0^{(q)}$*, respectively.*
2. *We have* $(w_p - v_q)(0) = 0$*, and the following limits are well defined and finite:*
$$\lim_{x\downarrow 0} \frac{(w_p - v_q)(x)}{W_q(x)}, \quad \lim_{x\uparrow 0} \frac{(w_p - v_q)(x)}{|x|}.$$
3. *There exist bounded,* $\mathcal{F}_{\tau_b^+\wedge\tau_0^-}$*-measurable functionals F and G such that*
$$w_p(x) = \mathbb{E}_x(F) \qquad \text{and} \qquad v_q(x) = \mathbb{E}_x(G), \quad x<0.$$

Then we have

$$\mathbf{n}\left(e^{-q\tau_0^-}(w_p - v_q)(X(\tau_0^-)); \tau_0^- < \tau_b^+\right) = \lim_{x\downarrow 0} \frac{(w_p - v_q)(x)}{W_q(x)} + \frac{\sigma^2}{2}\lim_{x\uparrow 0} \frac{(w_p - v_q)(x)}{|x|}$$
$$- \frac{1}{W_q(b)}\left((w_p - v_q)(b) - (p-q)\int_0^b W_q(b-y)w_p(y)dy\right).$$

Two key results.

Fix $b > 0$. Let us consider

$$\eta_0^- := \inf\{t > 0 : Y^b(t) < 0\}.$$

Theorem (Simplifying formula for the reflected process.)

Fix $p \geq 0$ and $b > 0$. Suppose $v_p : \mathbb{R} \to [0, \infty)$ and belongs to $\tilde{\mathcal{V}}_0^{(p)}$. Assume also that v_p is right-hand differentiable at b and $\int_{(-\infty,-1]} v_p(y+\theta)\Pi(d\theta) < \infty$ for all $0 \leq y \leq b$. Then, for $x \leq b$ and $q \geq 0$,

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-q\eta_0^-} v_p(Y^b(\eta_0^-)); \eta_0^- < \infty\right] \\
&\qquad = -\frac{W_q(x)}{W_q'(b+)}\left(v_p'(b+) - (p-q)\left[\int_0^b W_q'(b-y)v_p(y)dy + W_q(0)v_p(b)\right]\right) \\
&\qquad + v_p(x) - (p-q)\int_0^x W_q(x-y)v_p(y)dy.
\end{aligned}$$

Main results for X_r.

Theorem (Joint Laplace transform with killing)

For all $q, \theta \geq 0$, $a < 0 < b$, and $x \leq b$,

$$g(x,a,b,\theta) := \mathbb{E}_x\left(e^{-q\tau_b^+(r)-\theta R_r(\tau_b^+(r))}; \tau_b^+(r) < \tau_a^-(r)\right) = \frac{\mathcal{H}^a_{q,r}(x,\theta)}{\mathcal{H}^a_{q,r}(b,\theta)},$$

$$h(x,a,b,\theta) := \mathbb{E}_x\left(e^{-q\tau_a^-(r)-\theta R_r(\tau_a^-(r))}; \tau_a^-(r) < \tau_b^+(r)\right) = \mathcal{I}^a_{q,r}(x) - \frac{\mathcal{H}^a_{q,r}(x,\theta)}{\mathcal{H}^a_{q,r}(b,\theta)}\mathcal{I}^a_{q,r}(b),$$

where, for $y \in \mathbb{R}$,

$$\mathcal{H}^a_{q,r}(y,\theta) := r\int_0^{-a} e^{-\theta u}\left[W_a^{q,r}(y)\frac{W^{(q+r)}(u)}{W^{(q+r)}(-a)} - W_{-u}^{q,r}(y)\right]du + \frac{W_a^{q,r}(y)}{W^{(q+r)}(-a)},$$

$$\mathcal{I}_a^{q,r}(y) := Z_a^{q,r}(y) - W_a^{q,r}(y)\frac{Z^{(q+r)}(-a)}{W^{(q+r)}(-a)}.$$

Main results for X_r.

Theorem (Total discounted capital injections with killing)

For $a < 0 < b$, $q \geq 0$, and $x \leq b$,

$$\mathbb{E}_x\left(\int_0^{\tau_b^+(r)\wedge\tau_a^-(r)} e^{-qt} dR_r(t)\right) = \frac{\mathcal{H}^a_{q,r}(x,0)}{\mathcal{H}^a_{q,r}(b,0)} h^a_{q,r}(b) - h^a_{q,r}(x),$$

where, for $y \in \mathbb{R}$,

$$h^a_{q,r}(y) := \frac{r}{q+r}\left(\overline{Z}^{(q)}(y) + \frac{aZ^{(q+r)}(-a) + \overline{Z}^{(q+r)}(-a)}{W^{(q+r)}(-a)} W_a^{q,r}(y) - aZ_a^{q,r}(y) - \overline{Z}_a^{q,r}(y)\right),$$

Main results for Y_r^b.

Theorem (Joint Laplace transform)

Fix $a < 0 < b$, $q \geq 0$, and $\theta \geq 0$. For all $x \leq b$,

$$\mathbb{E}_x\left(e^{-q\eta_a^-(r)-\theta R_r^b(\eta_a^-(r))}; \eta_a^-(r) < \infty\right) = \mathcal{I}_{q,r}^a(x) - \frac{\mathcal{H}_{q,r}^a(x,\theta)}{\mathcal{H}_{q,r}^{a\prime}(b,\theta)}\mathcal{I}_{q,r}^{a\prime}(b),$$

Theorem (Total discounted dividends with killing)

For $a < 0 < b$ and $q \geq 0$, we have

$$\widehat{j}(x,a,b) := \mathbb{E}_x\left(\int_0^{\eta_a^-(r)} e^{-qt} dL_r^b(t)\right) = \begin{cases} \mathcal{H}_{q,r}^a(x,0)/\mathcal{H}_{q,r}^{a\prime}(b,0) & x \leq b, \\ \mathcal{H}_{q,r}^a(b,0)/\mathcal{H}_{q,r}^{a\prime}(b,0) + (x-b) & x > b. \end{cases}$$

Main results for Y_r^b.

Theorem (Total discounted capital injections with killing)

Suppose $q \geq 0$ and $a < 0 < b$. We have

$$\mathbb{E}_x\left(\int_0^{\eta_a^-(r)} e^{-qt} dR_r^b(t)\right) = \begin{cases} \frac{\mathcal{H}_{q,r}^a(x,0)}{\mathcal{H}_{q,r}^{a\prime}(b,0)} h_{q,r}^{a\prime}(b) - h_{q,r}^a(x) & x \leq b, \\ \frac{\mathcal{H}_{q,r}^a(b,0)}{\mathcal{H}_{q,r}^{a\prime}(b,0)} h_{q,r}^{a\prime}(b) - h_{q,r}^a(b) & x > b, \end{cases}$$

where $h_{q,r}^{a\prime}(y)$ is the (right-hand) derivatives of $h_{q,r}^a(y)$ with respect to y.

Periodic barrier strategies in the dual model.

- Let us consider X a spectrally positive Lévy process, as a model for the surplus of a company. Examples include discovery, commission-based, and mining companies.
- While a majority of the existing continuous-time models assume that dividends can be made at all times and instantaneously, in reality dividend decisions can only be made at some intervals.
- Let us consider the optimal dividend problem under the constraint that dividend payments can only be made at the jump times of an independent Poisson times.
- Avanzi et al. [2] solved the case of Brownian motion with i.i.d. hyperexponential jumps.
- We also study its extension with classical bail-outs so that the surplus is restricted to be nonnegative uniformly in time.

The optimal dividend problem with Poissonian dividend-decision times.

- We will assume that the dividend payments can only be made at the arrival times of a Poisson process $N^r = (N^r(t); t \geq 0)$ with intensity $r > 0$, which is independent of the Lévy process X.
- The set of dividend-decision times is denoted by $\mathcal{T}_r := (T(i); i \geq 0)$, where $T(i)$, for each $i \geq 0$, represents the i^{th} arrival time of the Poisson process N^r.
- In this setting, a strategy $\pi := (L^\pi(t); t \geq 0)$ is a nondecreasing, right-continuous, and $\mathbb{F}$-adapted process where the cumulative amount of dividends L^π admits the form

$$L^\pi(t) = \int_{[0,t]} \nu^\pi(s) dN^r(s), \qquad t \geq 0.$$

Here, for each $t \geq 0$, $\nu^\pi(t)$ represents the dividend payment at time t associated with the strategy π.

- The surplus process U^π after dividends are deducted is such that

$$U^\pi(t) := X(t) - L^\pi(t) = X(t) - \sum_{i=1}^{\infty} \nu^\pi(T(i)) 1_{\{T(i) \leq t\}}, \qquad 0 \leq t \leq \tau_0^\pi,$$

where

$$\tau_0^\pi := \inf\{t > 0 : U^\pi(t) < 0\}$$

is the corresponding ruin time.

The optimal dividend problem with Poissonian dividend-decision times.

- While the payment of dividends is allowed to cause immediate ruin, it cannot exceed the amount of surplus currently available. In other words, we also assume that

$$0 \leq \Delta L^{\pi}(T(i)) = \nu^{\pi}(T(i)) \leq U^{\pi}(T(i)-), \qquad \text{for } i \geq 1.$$

- Let $\mathcal{A}$ be the set of all admissible strategies that satisfy all the constraints described above. The problem is to maximize, for $q > 0$, the expected NPV of dividends paid until ruin associated with the strategy $\pi \in \mathcal{A}$, defined as

$$v_{\pi}(x) := \mathbb{E}_x\Big(\int_{[0,\tau_0^{\pi}]} e^{-qt} dL^{\pi}(t)\Big) = \mathbb{E}_x\Big(\int_{[0,\tau_0^{\pi}]} e^{-qt} \nu^{\pi}(t) dN^r(t)\Big), \quad x \geq 0.$$

Hence the problem is to compute the value function

$$v(x) := \sup_{\pi \in \mathcal{A}} v_{\pi}(x), \quad x \geq 0,$$

and obtain the optimal strategy π^* that attains it, if such a strategy exists.

Extension with classical bail-outs.

- We consider a version where the time horizon is infinity.
- The shareholders are required to inject capital to prevent the company from going bankrupt, with extra conditions on the dividend strategy described below.
- A strategy is a pair $\bar{\pi} := \left(L^{\bar{\pi}}(t), R^{\bar{\pi}}(t); t \geq 0\right)$ of nondecreasing, right-continuous, and $\mathbb{F}$-adapted processes where $L^{\bar{\pi}}$ is the cumulative amount of dividends and $R^{\bar{\pi}}$ is that of injected capital.
- The corresponding process is given by $U^{\bar{\pi}}(0-) := x$ and

$$U^{\bar{\pi}}(t) := X(t) - L^{\bar{\pi}}(t) + R^{\bar{\pi}}(t), \quad t \geq 0,$$

  and $(L^{\bar{\pi}}, R^{\bar{\pi}})$ must be chosen so that $U^{\bar{\pi}}$ stays nonnegative uniformly in time.
- In addition, we will assume that the cumulative amount of dividends can only occur at the arrival times of a Poisson process in $\mathcal{T}_r$, and so, in a similar way we have that $L^{\bar{\pi}}$ admits the form

$$L^{\bar{\pi}}(t) = \int_{[0,t]} \nu^{\bar{\pi}}(s) dN^r(s), \qquad t \geq 0,$$

  where $\nu^{\bar{\pi}}(t)$ represents the dividend payment at time t associated with the strategy $\bar{\pi}$.

Extension with classical bail-outs.

- Assuming that $\beta > 1$ is the cost per unit injected capital and $q > 0$ is the discount factor, we want to maximize

$$u_{\bar{\pi}}(x) := \mathbb{E}_x \left(\int_{[0,\infty)} e^{-qt} dL^{\bar{\pi}}(t) - \beta \int_{[0,\infty)} e^{-qt} dR^{\bar{\pi}}(t) \right), \quad x \geq 0,$$

over the set of all admissible strategies $\bar{\mathcal{A}}$ that satisfy all the constraints described above and

$$\mathbb{E}_x \left(\int_{[0,\infty)} e^{-qt} dR^{\bar{\pi}}(t) \right) < \infty.$$

- Hence the problem is to compute the value function

$$u(x) := \sup_{\bar{\pi} \in \bar{\mathcal{A}}} u_{\bar{\pi}}(x), \quad x \geq 0,$$

and obtain an optimal strategy $\bar{\pi}^*$ that attains it, if such a strategy exists.

Classical Case.

- Let us consider a barrier strategy, namely the payment of dividends is given by:

$$L_t^b = \sup_{0 \le s \le t} (X_s - b) \vee 0.$$

- And the controlled process by:

$$U_t^b = X_t - L_t^b.$$

- Now let us take

$$b = \begin{cases} (\overline{Z}^{(q)})^{-1}(-\psi'(0+)/q) & \psi'(0+) > 0, \\ 0 & \psi'(0+) \le 0. \end{cases}$$

- In this case the value function is given by

$$v_b(x) = \begin{cases} -\overline{Z}^{(q)}(b-x) - \frac{\psi'(0+)}{q} & \psi'(0+) > 0, \\ x & \psi'(0+) \le 0. \end{cases}$$

Theorem (Bayraktar, Kyprianou, Yamazaki)

The barrier strategy $\pi_b = \{L_t^b : t \ge 0\}$ is optimal over all admissible strategies.

Periodic barrier strategies.

- Our objective for the first problem is to show the optimality of the periodic barrier strategy, say π^b, with a suitable barrier level $b \geq 0$.
- At each Poissonian dividend-decision time, dividends are paid whenever the surplus process is above b and is pushed down so that the remaining surplus becomes b.
- The controlled process, which we formally construct below, is precisely the dual process of the *Parisian-reflected process*.
- Its value function is given by:

$$v_{b^*}(x) = -H^{(q,r)}(b^* - x) - \frac{Z^{(q,r)}(b^* - x)}{\Phi(q+r)}, \qquad \text{for } x \geq 0,$$

with

$$H^{(q,r)}(y) := \frac{r}{r+q}\left(\overline{Z}^{(q)}(y) + \frac{\psi'(0+)}{q}\right), \quad y \in \mathbb{R}.$$

Selection of the optimal barrier b^* .

- The barrier b^* (if strictly positive) is set so that the value function becomes "smooth" at the barrier, and is given as the unique solution to:

$$-\frac{1}{Z^{(q,r)}(b)}\frac{r}{r+q}\left(\overline{Z}^{(q)}(b)+\frac{\psi'(0+)}{q}\right)=\frac{1}{\Phi(q+r)}$$

- We have that $b^*>0$ if and only if

$$\psi'(0+)<-\frac{q}{r}\frac{r+q}{\Phi(q+r)}=:I_{r,q}.$$

- For the case in which

$$\psi'(0+)\geq I_{r,q}$$

holds, we will take the candidate optimal barrier as $b^*=0$; namely, the corresponding strategy *takes all the money and run* at the first Poissonian dividend-decision time.

- The threshold $I_{r,q}$ vanishes in the limit as $r\to\infty$. In other words, the criterion for $b^*=0$ converges to that in the classical case as the frequency of dividend-decision opportunities increases to infinity.
- On the other hand, as $r\to 0$, $I_{r,q}\to-\infty$, which means $b^*=0$ for small enough $r>0$. This suggests to take all the money and run if one needs to expect a long time until the next dividend-decision time.

Verification.

- We call a measurable function g *sufficiently smooth* if g is $C^1(0,\infty)$ (resp. $C^2(0,\infty)$) when X has paths of bounded (resp. unbounded) variation.
- We let $\mathcal{L}$ be the operator acting on a sufficiently smooth function g, defined by

$$\mathcal{L}g(x) := -\gamma g'(x) + \frac{\sigma^2}{2} g''(x) + \int_{(0,\infty)} [g(x+z) - g(x) - g'(x)z\mathbf{1}_{\{0<z<1\}}]\Pi(\mathrm{d}z).$$

- Then we have the following verification lemma:

Lemma (Verification lemma)

Suppose $\hat{\pi} \in \mathcal{A}$ is such that $v_{\hat{\pi}}$ is sufficiently smooth on $(0,\infty)$, and satisfies

$$(\mathcal{L} - q)v_{\hat{\pi}}(x) + r \max_{0 \le l \le x} \{l + v_{\hat{\pi}}(x-l) - v_{\hat{\pi}}(x)\} \le 0, \quad x > 0.$$

Then $v_{\hat{\pi}}(x) = v(x)$ for all $x \ge 0$ and hence $\hat{\pi}$ is an optimal strategy.

The optimal dividend problem with Poissonian dividend-decision times.

Theorem

The periodic barrier strategy π^{b^*} is optimal and the value function is given by $v(x) = v_{b^*}(x)$ for all $0 \leq x < \infty$.

As $r \to \infty$ the optimal barrier b^* as well as the value function v_{b^*} converge to those in the classical case (assuming $\mathbb{E}X_1 = -\psi'(0+) > 0$): $\tilde{b}^* := (\overline{Z}^{(q)})^{-1}(-\psi'(0+)/q)$ and

$$\tilde{v}(x) := -\overline{Z}^{(q)}(\tilde{b}^* - x) - \frac{\psi'(0+)}{q}, \quad x \geq 0,$$

as obtained in Bayraktar et al. [4].

Classical Case.

- In this case we consider the doubly ref!ected Lévy process with upper barrier $\tilde{b} > 0$ and lower barrier 0 of the form

$$U_t^{\tilde{b}} = X_t - L_t^{\tilde{b}} + R_t^0 \qquad t \geq 0.$$

- Now let us take

$$\tilde{b} = (Z^{(q)})^{-1}(\beta).$$

- In this case the value function is given by

$$v_{\tilde{b}}(x) = -\overline{Z}^{(q)}(\tilde{b} - x) - \frac{\psi'(0+)}{q}, \quad x \geq 0.$$

Theorem (Bayraktar, Kyprianou, Yamazaki)

The strategy $\pi_{\tilde{b}} = \{L_t^{\tilde{b}}, R_t^0 : t \geq 0\}$ *is optimal over all admissible strategies.*

Periodic-classic strategies.

- For the second problem, we want to show the optimality of an extension of the above strategy with additional classical reflection (capital injection) below at 0, say $\bar{\pi}^{b^\dagger}$, with a suitable Parisian reflection level $b^\dagger \geq 0$.
- Namely, dividends are paid whenever the surplus process is above $b^\dagger$ at dividend-decision times, while it is pushed upward by capital injection whenever it attempts to down-cross zero.
- The controlled process is the dual of the classical-Parisian reflected Lévy process.
- Its value function is given by:

$$v_{b^*}(x) = -H^{(q,r)}(b^\dagger - x) - \frac{Z^{(q,r)}(b^\dagger - x)}{\Phi(q+r)}, \qquad \text{for } x \geq 0.$$

Extension with classical bail-outs.

- In this case the optimal threshold $b^{\dagger} > 0$ is given by the unique solution to the following equation:

$$-\frac{1}{Z^{(q,r)\prime}(b)}\left(\frac{rZ^{(q)}(b)}{r+q} - \beta\right) = \frac{1}{\Phi(q+r)}$$

- And we have the following verification lemma:

Lemma (Verification lemma)

Suppose $\hat{\pi}$ is an admissible dividend strategy such that $u_{\hat{\pi}}$ is sufficiently smooth on $(0,\infty)$ and differentiable at zero (i.e. $u'_{\hat{\pi}}(0) = \beta$), and satisfies

$$\begin{aligned} (\mathcal{L}-q)u_{\hat{\pi}}(x) + r\max_{0\le l\le x}\{l + u_{\hat{\pi}}(x-l) - u_{\hat{\pi}}(x)\} &\le 0, \quad x > 0, \\ u'_{\hat{\pi}}(x) &\le \beta, \quad x > 0, \\ \inf_{x\ge 0} u_{\hat{\pi}}(x) &> -m, \quad \text{for some } m > 0. \end{aligned}$$

Then $u_{\hat{\pi}}(x) = u(x)$ for all $x \ge 0$ and hence $\hat{\pi}$ is an optimal strategy.

Extension with classical bail-outs.

Theorem

The periodic barrier strategy $\bar{\pi}^{b^\dagger}$ *with classical reflection from below at* 0 *is optimal and the value function is* $u(x) = u_{b^\dagger}(x)$ *for all* $0 \leq x < \infty$.

Similarly, as $r \to \infty$, the optimal barrier $b^\dagger$ as well as the value function $u_{b^\dagger}$ are converge to those in the classical case: $\tilde{b}^\dagger := (Z^{(q)})^{-1}(\beta)$ and

$$\tilde{u}(x) := -\overline{Z}^{(q)}(\tilde{b}^\dagger - x) - \frac{\psi'(0+)}{q}, \quad x \geq 0;$$

see Bayraktar et al. [4].

Numerical results.

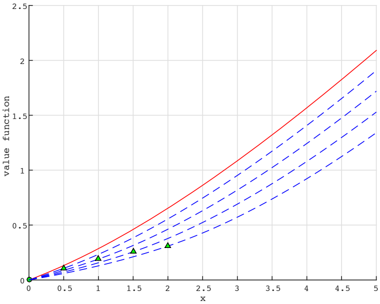
- We assume X is the spectrally positive version of the *phase-type* Lévy process (with a Brownian motion).
- More specifically, for some $c \in \mathbb{R}$ and $\sigma > 0$,

$$X(t) - X(0) = -ct + 0.2B(t) + \sum_{n=1}^{N(t)} Z_n, \quad 0 \leq t < \infty,$$

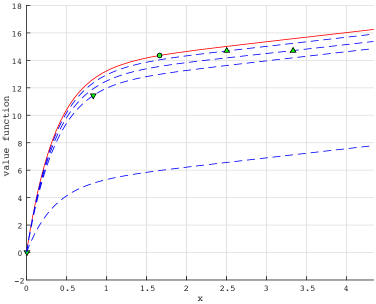
where $B = (B(t); t \geq 0)$ is a standard Brownian motion, $N = (N(t); t \geq 0)$ is a Poisson process with arrival rate 2, and $Z = (Z_n; n = 1, 2, \ldots)$ is an i.i.d. sequence of phase-type-distributed random variables with representation $(6, \alpha, T)$, or equivalently the first absorption time in a continuous-time Markov chain consisting of a single absorbing state and 6 transient states with its initial distribution α and transition matrix T.

- The processes B, N, and Z are assumed mutually independent.

Numerical results.



$x \mapsto v_{b^*}(x)$ and $x \mapsto v_b(x)$



$x \mapsto v_{b^*}(x)$ and $x \mapsto v_b(x)$

Figure: (Left) The corresponding value function $v_{b^*}(x)$ (solid) along with suboptimal expected NPVs v_b (dotted) for $b = 0, b^*/2, 3b^*/2, 2b^*$ for **Case 1,** $c = 0.5$ and $b = 0.5, 1, 1.5, 2$ for **Case 2,** $c = 2$. The values at b^* are indicated by circles whereas those at the suboptimal barriers $b > b^*$ (resp. $b < b^*$) are indicated by up-pointing (resp. down-pointing) triangles.

Numerical results.

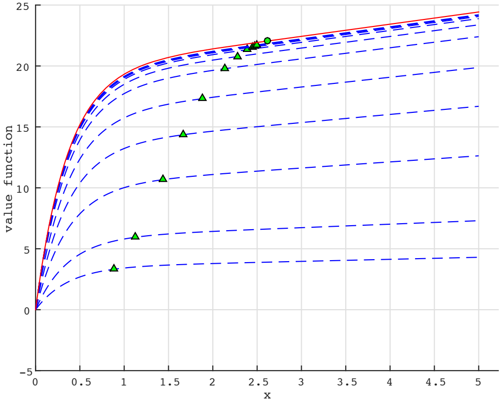
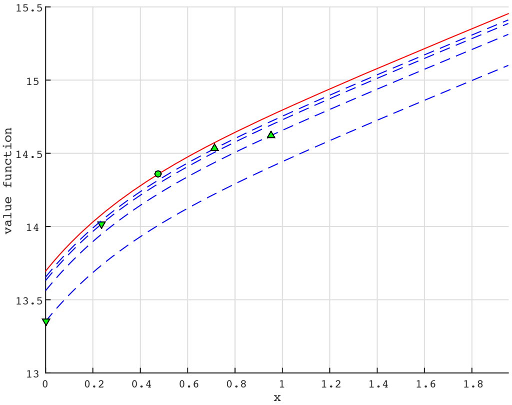


Figure: The value functions v_{b^*} (dotted) for $r = 0.01$, 0.02, 0.05, 0.1, 0.2, 0.5, 1, 2, 3, 4, 5 along with the value function $\tilde{v}$ in the classical case (solid). The up-pointing triangles show the points at b^* of v_{b^*}; the circle shows the point at $\tilde{b}^*$ of $\tilde{v}$.

Numerical results.



$$x \mapsto u_{b^\dagger}(x)$$

Figure: (Left) The corresponding value function $u_{b^\dagger}(x)$ (solid) along with suboptimal expected NPVs u_b for $b = 0, b^\dagger/2, 3b^\dagger/2, 2b^\dagger$ (dotted). The values at $b^\dagger$ are indicated by circles whereas those at $b > b^\dagger$ (resp. $b < b^\dagger$) for suboptimal expected NPVs are indicated by up-pointing (resp. down-pointing) triangles.

Numerical results.

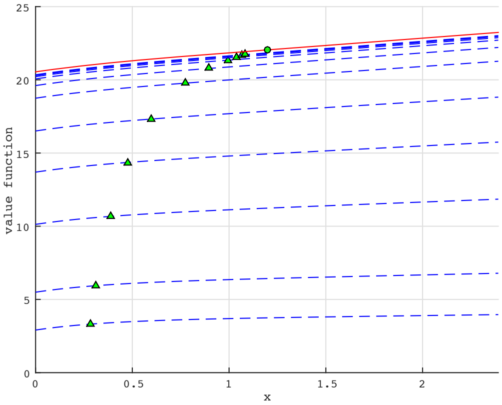


Figure: The value functions $u_{b^\dagger}$ (dotted) for $r = 0.01, 0.02, 0.05, 0.1, 0.2, 0.5, 1, 2, 3, 4, 5$ along with the value function $\tilde{u}$ in the classical case (solid). The up-pointing triangles show the points at $b^\dagger$ of $u_{b^\dagger}$; the circle shows the point at $\tilde{b}^\dagger$ of $\tilde{u}$.

Conclusion

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Conclusion

Thank for your attention!