Continuous state branching processes in a Lévy random environment.

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based on joint works with Sandra Palau & Charline Smadi

CB-processes.

A continuous-state branching process (or CB-process) is a non-negative valued strong Markov process with probabilities $(\mathbb{P}_x,\ x \geq 0)$ such that for any $x, y \geq 0$, $\mathbb{P}_{x+y}$ is equal in law to the convolution of $\mathbb{P}_x$ and $\mathbb{P}_y$.

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More precisely, a CB-process $Y = (Y_t, t \geq 0)$ is a Markov process taking values in $[0, \infty]$, where 0 and ∞ are two absorbing states, and satisfying the branching property.

In particular,

$$\mathbb{E}_x\left[e^{-\lambda Y_t}\right] = \exp\{-x u_t(\lambda)\}, \qquad \text{for } \lambda \geq 0,$$

for some function $u_t(\lambda)$.

The function $u_t(\lambda)$ is determined by the integral equation

$$\int_{u_t(\lambda)}^{\lambda} \frac{1}{\psi(u)} \mathrm{d}u = t$$

where ψ (branching mechanism of Y) satisfies the Lévy-Khincthine formula

$$\psi(\lambda) = -a\lambda + \gamma^2\lambda^2 + \int_{(0,\infty)} \big(e^{-\lambda x} - 1 + \lambda x \mathbf{1}_{\{x<1\}}\big)\mu(\mathrm{d}x),$$

where $a \in \mathbb{R}$, $\gamma \geq 0$ and μ is a σ-finite measure such that

$$\int_{(0,\infty)} \big(1 \wedge x^2\big)\mu(\mathrm{d}x) < \infty.$$

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Observe $\mathbb{E}_x[Y_t] = xe^{-\psi'(0^+)t}$. Hence, in respective order, a CB-process is called supercritical, critical or subcritical accordingly as $\psi'(0^+) < 0$, $\psi'(0^+) = 0$ or $\psi'(0^+) > 0$.

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A CB-process Y with branching mechanism ψ has a finite time extinction almost surely if and only if

$$\int^{\infty} \frac{\mathrm{d}u}{\psi(u)} < \infty \quad \text{and} \quad \psi'(0+) \geq 0.$$

A CB-process can also be defined as the unique non-negative strong solution of the stochastic differential equation

$$Y_t = Y_0 + a\int_0^t Y_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Y_s}\mathrm{d}B_s + \int_0^t \int_{(0,1)} \int_0^{Y_{s-}} z\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Y_{s-}} zN(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where $B = (B_t, t \geq 0)$ is a standard Brownian motion, N is a Poisson random measure independent of B, with intensity $\mathrm{d}s \otimes \mu(\mathrm{d}z) \otimes \mathrm{d}u$ and $\tilde{N}$ is its compensated version.

CB-process in a Lévy random environment

We introduce a continuous state branching process in a Lévy random environment (CBLRE) as the unique non-negative strong solution of the stochastic differential equation

$$
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Z_t = & Z_0 + a \int_0^t Z_s \mathrm{d}s + \int_0^t \sqrt{2\gamma^2 Z_s} \mathrm{d}B_s \\
& + \int_0^t \int_{(0,1)} \int_0^{Z_{s-}} z \tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{Z_{s-}} z N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),
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where

$$\begin{aligned} S_t = \alpha t + \sigma B_t^{(e)} &+ \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_0^t \int_{\mathbb{R}\setminus(-1,1)} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

with $\alpha \in \mathbb{R}$ and $\sigma \geq 0$, $B^{(e)} = (B_t^{(e)}, t \geq 0)$ is a standard Brownian motion and $N^{(e)}(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure in $\mathbb{R}_+ \times \mathbb{R}$ independent of $B^{(e)}$ with intensity $\mathrm{d}s\pi(\mathrm{d}y)$, $\widetilde{N}^{(e)}$ its compensated version and π is a σ-finite measure satisfying

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When $|\psi'(0+)| < \infty$, we define the auxiliary process

$$K_t = \mathbf{m}t + \sigma B_t^{(e)} + \int_0^t \int_{(-1,1)} v\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}v) + \int_0^t \int_{\mathbb{R}\setminus(-1,1)} vN^{(e)}(\mathrm{d}s, \mathrm{d}v),$$

where

$$\mathbf{m} = \alpha - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v) - \psi'(0+).$$

Let $C^2(\mathbb{R}_+)$ and $D(\mathbb{R}_+)$ be the sets of functions with continues first and second derivatives and the set of càdlàg functions, respectively.

Theorem

The previous stochastic differential equation has a unique non-negative strong solution. The process $Z = (Z_t, t \geq 0)$ is a Markov process and its infinitesimal generator $\mathcal{L}$ satisfies, for every $f \in C^2(\mathbb{R}_+)$,

$$\begin{aligned}\mathcal{A}f(x) &= (\mathbf{m} + a)xf'(x) + \left(\gamma^2 x + \frac{\sigma^2}{2}x^2\right) f''(x) \\ &\quad + x\int_{(0,\infty)} \Big(f(x+z) - f(x) - zf'(x)\mathbf{1}_{\{z<1\}}\Big)\Lambda(\mathrm{d}\, z) \\ &\quad + \int_{\mathbb{R}} \Big(f(xe^z) - f(x) - x(e^z - 1)f'(x)\mathbf{1}_{\{|z|<1\}}\Big)\pi(\mathrm{d}\, z).\end{aligned}$$

Furthermore, the process Z, conditionally on K, satisfies the branching property and for $|\psi'(0+)| < \infty$, we have for every $t > 0$

$$\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t}\right\}\Big| K\right] = \exp\left\{-z v_t(0, \lambda, K)\right\} \qquad a.s.,$$

where for every $(\lambda, \delta) \in (\mathbb{R}_+, D(\mathbb{R}_+))$, $v_t : s \in [0, t] \mapsto v_t(s, \lambda, \delta)$

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$$\frac{\partial}{\partial s} v_t(s, \lambda, \delta) = e^{\delta_s} \psi_0 (v_t(s, \lambda, \delta) e^{-\delta_s}),$$

with $\psi_0(\lambda) = \psi(\lambda) - \lambda\psi'(0+)$, *and* ψ *is the branching mechanism of the underlying CB-process.*

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Idea of the proof: First, we fix $n \geq 1$ and prove the existence of a positive weak solution of the SDE

$$Z_t^n = Z_0^n + a\int_0^t (Z_s^n \wedge n)\mathrm{d}s + \int_0^t \sqrt{2\gamma^2(Z_s^n \wedge n)}\mathrm{d}B_s + \int_0^t (Z_s^n \wedge n)\mathrm{d}S_s \quad (1)$$

$$+ \int_0^t \int_{(0,1)} \int_0^{(Z_s^n \wedge n)-} (z \wedge n)\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t \int_{[1,\infty)} \int_0^{(Z_s^n \wedge n)-} (z \wedge n)N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u).$$

Similar techniques as those used in Dawson & Li (2012) provides the pathwise uniqueness of (1). Basically, we take Z^n and $Z^{n,\prime}$ two solutions of (1) and prove that the expectation of $|Z^n - Z^{n,\prime}|$ equals 0.

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For $m \geq 1$ let $\tau_m = \inf\{t \geq 0 : Z_t^m \geq m\}$. By a localization argument, we may construct

$$Z_t = \begin{cases} Z_t^m & \textit{if} \quad t < \tau_m \\ \infty & \textit{if} \quad t \geq \lim\limits_{m\to\infty} \tau_m \end{cases}$$

which is a weak solution to our original equation.

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which is a weak solution to our original equation.

Finally, let Z' and Z'' be two solutions of our original equation and also consider $\tau'_m = \inf\{t \geq 0 : Z'_t \geq m\}$, $\tau''_m = \inf\{t \geq 0 : Z''_t \geq m\}$ and define $\tau_m = \tau'_m \wedge \tau''_m$.

Then, Z' and Z'' satisfy (1) on $[0, \tau_m)$, so they are indistinguishable on $[0, \tau_m)$. When $\tau_\infty = \lim_{m\to\infty} \tau_m < \infty$, Z' or Z'' have a jump of infinity size in τ_∞. This jump comes from an atom of N, so that both processes have it and thus Z' and Z'' are indistinguishable. Then, the strong solution to our original equation follows.

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Let $\widetilde{Z}_t = Z_t e^{-K_t}$ and $v_t(s, \lambda, K)$ is differentiable with respect to the variable s, non-negative and such that $v_t(t, \lambda, K) = \lambda$ for all $\lambda \geq 0$.

Applying Itô's formula one obtain that $\exp\{-\widetilde{Z}_t v_t(s,\lambda,K)\}$ conditionally on K is a martingale if and only if for every $t \geq 0$,

$$\frac{\partial}{\partial s} v_t(s,\lambda,K) = -a v_t(s,\lambda,K) + \gamma^2 (v_t(s,\lambda,K))^2 e^{-Ks}$$
$$+ e^{K_s} \int_0^\infty \left(e^{-e^{-K_s} v_t(s,\lambda,K) z} - 1 + e^{-K_s} v_t(s,\lambda,K) z \mathbf{1}_{\{z<1\}} \right) \mu(\mathrm{d}z).$$

In the case when $|\psi'(0+)| = -\infty$, the auxiliary process is almost the same. The only thing that changes is the drift which is given as follows

$$\mathbf{m} = \alpha - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v).$$

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In this case, one can deduce that $v_t(s, \lambda, K)$ is the unique solution to

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In this case, the process Z, conditionally on K, satisfies

$$\mathbb{E}_z\Big[\exp\Big\{-\lambda Z_t e^{-K_t}\Big\}\Big|K\Big] = \exp\Big\{-zv_t(0, \lambda, K)\Big\} \qquad a.s.$$

Examples

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Then,

$$\mathbb{E}_z\left[\exp\left\{ -\lambda Z_t e^{-K_t} \right\} \middle| K \right] = \exp\left\{ -z\lambda^{e^{-t}} \exp\left\{ \int_0^t e^{-s} K_s \mathrm{d}s \right\} \right\} \qquad a.s.,$$

which implies that

$$\mathbb{P}_z\left(Z_t > 0 \middle| K \right) = 1, \qquad t > 0.$$

Observe that

$$\mathbb{E}_z\left[Z_t e^{-K_t}\exp\left\{-\lambda Z_t e^{-K_t}\right\}\Big|K\right] = ze^{-t}\lambda^{e^{-t}-1}$$
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This implies

$$\mathbb{E}_z[Z_t] = \infty, \qquad t > 0.$$

Moreover, when K is just a Brownian motion with drift, the r.v. $\int_0^t e^{-s}K_s\mathrm{d}s$ is Gaussian with mean $(\alpha - \frac{\sigma^2}{2})(1 - e^{-t} - te^{-t})$ and variance $\frac{\sigma^2}{2}(1 - 4e^{-t} - 3e^{-2t})$.

Feller's diffusion

If $a = \mu(0, \infty) = 0$, the CBBRE is given by

$$Z_t = Z_0 + \alpha \int_0^t Z_s \, ds + \sigma \int_0^t Z_s \, dS_s + \int_0^t \sqrt{2\gamma^2 Z_s} \, dB_s.$$

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The above is equivalent to the strong solution of the SDE

$$\begin{aligned} \mathrm{d}Z_t =& \frac{\sigma^2}{2} Z_t \mathrm{d}t + Z_t \mathrm{d}K_t + \sqrt{2\gamma^2 Z_s} \mathrm{d}B_s \\ \mathrm{d}K_t =& \alpha \mathrm{d}t + \sigma \mathrm{d}W_t, \end{aligned}$$

which looks as the branching diffusion in random environment studied by Böinghoff and Hutzenthaler (2011).

Stable case. Here, the branching mechanism is of the form

$$\psi(\lambda) = -a\lambda + c_\beta \lambda^{\beta+1}, \qquad \lambda \geq 0,$$

for some $\beta \in (-1,0) \cup (0,1)$, $a \in \mathbb{R}$, and

$$\begin{cases} c_\beta < 0 & \text{if } \beta \in (-1,0), \\ c_\beta > 0 & \text{if } \beta \in (0,1). \end{cases}$$

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Under this assumption, the process Z satisfies the following stochastic differential equation

$$Z_t = Z_0 + a\int_0^t Z_s \mathrm{d}s + \int_0^t Z_{s-}\mathrm{d}S_s + \int_0^t \int_0^\infty \int_0^{Z_{s-}} z\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$$

and

$$\widehat{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) = \begin{cases} N(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (-1,0), \\ \widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) & \text{if } \beta \in (0,1), \end{cases}$$

where N is an independent Poisson random measure with intensity

$$\frac{c_\beta \beta(\beta+1)}{\Gamma(1-\beta)} \frac{1}{z^{2+\beta}} \mathrm{d}s\mathrm{d}z\mathrm{d}u,$$

In this case, we note

$$\psi'(0+) = \begin{cases} -\infty & \text{if } \beta \in (-1, 0), \\ -a & \text{if } \beta \in (0, 1). \end{cases}$$

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We use in both cases the backward differential equation of Theorem 1 and observe that it satisfies

$$\frac{\partial}{\partial s} v_t(s,\lambda,\delta) = -a v_t(s,\lambda,\delta) + c_\beta v_t^{\beta+1}(s,\lambda,\delta) e^{-\beta\delta_s}.$$

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Therefore,

$$v_t(s,\lambda,\delta) = e^{as}\left((\lambda e^{at})^{-\beta} + \beta c_\beta \int_s^t e^{-\beta(\delta_u + au)} \mathrm{d}u\right)^{-1/\beta}.$$

Implying the following a.s. identity

$$\mathbb{E}_z\left[\exp\left\{-\lambda Z_t e^{-K_t}\right\}\Big|K\right] = \exp\left\{-z\left(\lambda^{-\beta} + \beta c_\beta \int_0^t e^{-\beta K_u} \mathrm{d}u\right)^{-1/\beta}\right\}.$$

Long-term behaviour

Similarly to the case of CB-processes, there are three events which are of immediate concern for the process Z, *explosion*, *absorption* and *extinction*.

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Recall that $\psi'(0+) \in [-\infty, \infty)$, and that whenever $|\psi'(0+)| < \infty$, we write

$$\mathbf{m} = \alpha - \psi'(0+) - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^v - 1 - v)\pi(\mathrm{d}v).$$

where

$$\psi'(0+) = -a - \int_{\{x>1\}} x\mu(\mathrm{d}x).$$

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Proposition

Assume $|\psi'(0+)| < \infty$, then a CBPBRE Z with branching mechanism ψ satisfies

$$\mathbb{P}_z(Z_t < \infty) = 1, \qquad \textit{for all} \quad t > 0.$$

Stable case with $\beta \in (-1, 0)$.

Recall that in this case $\psi(u) = -au + c_\beta u^{\beta+1}$, where $a \in \mathbb{R}$ and c_β is a negative constant.

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From straightforward computations, we get $\psi'(0+) = -\infty$

Stable case with $\beta \in (-1, 0)$.

Recall that in this case $\psi(u) = -au + c_\beta u^{\beta+1}$, where $a \in \mathbb{R}$ and c_β is a negative constant.

From straightforward computations, we get $\psi'(0+) = -\infty$

From the Laplace transform of $\tilde{Z}$ (taking λ goes to 0), we deduce

$$\mathbb{P}_z\Big(Z_t < \infty \Big| K\Big) = \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta(K_u + au)} \mathrm{d}u\right)^{-1/\beta}\right\} \qquad \text{a.s.,}$$

implying

$$\mathbb{P}_z\Big(Z_t = \infty \Big| K\Big) = 1 - \exp\left\{-z\left(\beta c_\beta \int_0^t e^{-\beta(K_u + au)} \mathrm{d}u\right)^{-1/\beta}\right\} > 0.$$

Neveu case.

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By taking λ goes to 0 in the Laplace exponent of $\tilde{Z}$, one can see that the process is conservative conditionally on the environment, i.e.

$$\mathbb{P}_z(Z_t < \infty | K) = 1,$$

for all $t \in (0, \infty)$ and $z \in [0, \infty)$.

Proposition

Assume that $|\psi'(0+)| < \infty$. *Let* $(Z_t, t \geq 0)$ *be a CBPBRE with branching mechanism given by* ψ.

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i) *If* K *drifts to* $-\infty$, *then* $\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \middle| K\right) = 1$, *a.s.*

Proposition

Assume that $|\psi'(0+)| < \infty$. Let $(Z_t, t \geq 0)$ be a CBPBRE with branching mechanism given by ψ.

i) *If K drifts to $-\infty$, then $\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \middle| K\right) = 1$, a.s.*

ii) *If K oscillates, then $\mathbb{P}_z\left(\liminf_{t\to\infty} Z_t = 0 \middle| K\right) = 1$, a.s.*

Moreover if $\gamma > 0$ then

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0 \middle| K\right) = 1, a.s.$$

Proposition

iii) *If* K *drifts to* $+\infty$ *and*

$$\int^{\infty} x\ln(x)\,\mu(\mathrm{d}x) < \infty,$$

then $\mathbb{P}_z\left(\liminf_{t\to\infty} Z_t > 0 \middle| K\right) > 0$ *a.s., and there exists a non-negative finite r.v.* W *such that*

$$Z_t e^{-K_t} \underset{t\to\infty}{\longrightarrow} W, \; a.s \qquad and \qquad \{W = 0\} = \left\{\lim_{t\to\infty} Z_t = 0\right\}.$$

Moreover if $\gamma > 0$*, we have*

$$\mathbb{P}_z\left(\lim_{t\to\infty} Z_t = 0\right) \geq \left(1 + \frac{z\sigma^2}{\gamma^2}\right)^{-\frac{2\mathbf{m}}{\sigma^2}}.$$

It is important to note that in the Feller and stable cases, i.e. $\psi(u) = -au + c_\beta u^{\beta+1}$ for $\beta \in (0,1]$, one can deduce directly that

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The latter probability can be computed explicitly in some specific cases.

Neveu case. Recall that the Neveu CBPBRE process Z satisfies

$$\mathbb{P}(0 < Z_t < \infty) = 1,$$

for all $t \in (0, \infty)$ and $z \in (0, \infty)$.

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On the one hand, we have that

$$\lim_{t\to\infty} \int_0^t e^{-s} K_s \mathrm{d}s = \sigma \int_0^\infty e^{-s} dK_s + \alpha - \frac{\sigma^2}{2},$$

exists whenever $\mathbb{E}[\log |K_1|] < \infty$. In the case where K is a Brownian motion, the latter r.v. is Gaussian with mean $\alpha - \frac{\sigma^2}{2}$ and variance $\frac{\sigma^2}{2}$.

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exists whenever $\mathbb{E}[\log |K_1|] < \infty$. In the case where K is a Brownian motion, the latter r.v. is Gaussian with mean $\alpha - \frac{\sigma^2}{2}$ and variance $\frac{\sigma^2}{2}$. Hence,

$$\mathbb{E}_z \left[\exp\left\{ -\lambda \lim_{t\to\infty} Z_t e^{-K_t} \right\} \middle| K \right] = \exp\left\{ -z \exp\left\{ \int_0^\infty e^{-s} K_s \mathrm{d}s \right\} \right\}.$$

implying

$$\mathbb{P}_z \left(\lim_{t\to\infty} Z_t e^{-K_t} = 0 \right) = \mathbb{E}\left[\exp\left\{ -z \exp\left\{ \int_0^\infty e^{-s} K_s \mathrm{d}s \right\} \right\} \right].$$

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Let $(Z_t, t \geq 0)$ *be the stable CBLRE with index* $\beta \in (-1, 0)$ *and* $Z_0 = z > 0$.

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i) *Subcritical-explosion. If* $\phi'_K(0+) < 0$, *then there exist* $c_1(z) > 0$ *such that*

$$\lim_{t\to\infty} \mathbb{P}_z(Z_t < \infty) = c_1(z).$$

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ii) *Critical-explosion. If* $\phi'_K(0+) = 0$, *then then there exist* $c_2(z) > 0$ *such that*

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iii) *Supercritical-explosion. If $\phi'_K(0+) > 0$ then there exist $c_3(z) > 0$*

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\phi_K(\tau)}\mathbb{P}_z(Z_t < \infty) = c_3(z),$$

where τ is the value at which ϕ_K attains its minimum.

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ii) *(Critical case) If* $\phi'_K(0+) = 0$, *then there exist* $c_5(z) > 0$ *such that*

$$\lim_{t\to\infty} \sqrt{t}\mathbb{P}_z(Z_t > 0) = c_5(z).$$

Theorem

iii) *(Weakly subcritical) If* $\phi'_K(0+) = 0$ *and* $\phi'_K(1) > 0$*, then there exist* $c_6(z) > 0$ *such that*

$$\lim_{t\to\infty} t^{\frac{3}{2}} e^{\phi_K(\tau)} \mathbb{P}_z(Z_t > 0) = c_6(z),$$

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iv) *(Intermediately subcritical) If* $\phi'_K(0+) = 0$ *and* $\phi'_K(1) = 0$, *then there exist* $c_7 > 0$ *such that*

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v) *(Strongly subcritical)If* $\phi'_K(0+) = 0$ *and* $\phi'_K(1) < 0$ *(+ some moments conditions), then there exist* $c_7 > 0$ *such that*

$$\lim_{t\to\infty} e^{t\phi_K(1)} \mathbb{P}_z(Z_t > 0) = zc_8.$$