# Refracted-Reflected Spectrally one-sided Lévy Processes

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# Reflected & Refracted Lévy Processes

- Reflected Lévy processes: $U_t := X_t - L_t$ where

$$L_t := \sup_{0 \le t' \le t} (X_{t'} - b) \vee 0, \quad t \ge 0.$$

- Refracted Lévy processes (Kyprianou and Loeffen, 2009)
  - A strong Markov process given by the unique strong sol'n to the SDE

$$\mathrm{d}A_t = \mathrm{d}X_t - \delta \mathbf{1}_{\{A_t > b\}} \mathrm{d}t, \quad t \ge 0.$$

  - Namely, $A$ progresses like $X$ below the boundary $b$ while it does like

$$Y_t := X_t - \delta t, \quad t \ge 0,$$

above $b$.

# Refracted-Reflected Lévy Processes

- Refracted at an upper boundary $b$, and
- Reflected at a lower boundary $0$.

Step 0 Set $V_{0-} = x$. If $x \geq 0$, then set $\underline{\tau} := 0$ and go to **Step 1**. Otherwise, set $\overline{\tau} := 0$ and go to **Step 2**.

Step 1 Let $\{\widetilde{A}_t; t \geq \underline{\tau}\}$ be the refracted Lévy process (with refraction level $b$) that starts at the time $\underline{\tau}$ at the level $x$, and $\overline{\tau} := \inf\{t > \underline{\tau} : \widetilde{A}_t < 0\}$. Set $V_t = \widetilde{A}_t$ for all $\underline{\tau} \leq t < \overline{\tau}$. Then go to **Step 2**.

Step 2 Let $\{\widetilde{U}_t; t \geq \overline{\tau}\}$ be the Lévy process reflected at the lower boundary $0$ that starts at time $\overline{\tau}$ at $0$, and $\underline{\tau} := \inf\{t > \overline{\tau} : \widetilde{U}_t > b\}$. Set $V_t = \widetilde{U}_t$ for all $\overline{\tau} \leq t < \underline{\tau}$. Then go to **Step 1**.

# Decomposition

We have a decomposition:

$$V_t = X_t + R_t - L_t, \quad t \geq 0.$$

In particular, we have

$$L_t = \delta \int_0^t 1_{\{V_s > b\}} \mathrm{d}s, \quad t \geq 0,$$

and for the case of bounded variation

$$R_t = \sum_{t \geq 0 : V_{t-} + \Delta X_t < 0} |V_{t-} + \Delta X_t| \quad t \geq 0.$$

# Applications

- M/G/1 queues with abandonments
  - (reflection) The length of queues does not go negative.
  - (refraction) When the queue is long, some people may decide not to line up.
- Optimal dividends with capital injection
  - (reflection) Pay dividends at a constant rate $\delta$ when it is above $b$.
  - (refraction) Must inject capital to prevent it from going below zero.
- Optimal dividends with both singular and absolutely continuous control
  - (reflection) Pay dividends at a constant rate $\delta$ when it is above $a$.
  - (refraction) Pay dividends so that it does not go above $b(> 0)$.

# Objective

We compute fluctuation identities – with $T_a^+ := \inf\{t > 0 : V_t > a\}$,

- Resolvents:

$$\mathbb{E}_x\Big(\int_0^{T_a^+} e^{-qt} 1_{\{V_t \in B\}} \mathrm{d}t\Big)$$

- expected NPVs (net present value) of total discounted $L$ and $R$:

$$\mathbb{E}_x\Big(\int_0^{T_a^+} e^{-qt} \mathrm{d}L_t\Big) \quad \text{and} \quad \mathbb{E}_x\Big(\int_{[0,T_a^+]} e^{-qt} \mathrm{d}R_t\Big).$$

- Laplace transform of occupation time:

$$\mathbb{E}_x\left(e^{-pT_a^+ - q\int_0^{T_a^+} 1_{\{V_s < b\}} \mathrm{d}s}\right) \quad \text{and} \quad \mathbb{E}_x\left(e^{-pT_a^+ - q\int_0^{T_a^+} 1_{\{V_s > b\}} \mathrm{d}s}\right).$$

## Usual Tricks

Define

- The drift-changed process $Y_t := X_t - \delta t$.
- The reflected process $U$: $X$ reflected at lower level $0$.

Almost surely, under $\mathbb{P}_x$ for any $x \in \mathbb{R}$, we have

- $V_t = U_t$ on $0 \leq t \leq T_b^+$,
- $V_t = Y_t$ on $0 \leq t < T_b^-$ and $V_{T_b^- -} + \Delta X_{T_b^-} = Y_{\tau_b^-}$.

In view of this, we use the fluctuation theory written in terms of the scale function of $Y$ and $U$ together with the strong Markov property of $V$.

# SN Lévy Processes and Laplace Exponents

- Given a SN Lévy process $X = \{X_t; t \geq 0\}$, the Laplace exponent is

$$\psi(\theta) := \log \mathbb{E}[e^{\theta X_1}] = \gamma\theta + \frac{\sigma^2}{2}\theta^2 + \int_{(-\infty,0)} (e^{\theta z} - 1 - \theta z \mathbf{1}_{\{z>-1\}})\Pi(\mathrm{d}z), \quad \theta \geq 0.$$

- $\Pi$ is a Lévy measure such that $\int_{(-\infty,0)}(1 \wedge z^2)\Pi(\mathrm{d}z) < \infty$.
- It has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} |z| \, \Pi(\mathrm{d}z) < \infty$.
- For the case of bounded variation, we can write

$$\psi(\theta) = \tilde{\gamma}\theta + \int_{(-\infty,0)} (e^{\theta z} - 1)\Pi(\mathrm{d}z), \quad \theta \geq 0,$$

with $\tilde{\gamma} := \gamma - \int_{(-1,0)} z \, \Pi(\mathrm{d}z)$.

# Scale Functions

- We use $W^{(q)}$ and $\mathbb{W}^{(q)}$ for the scale functions of $X$ and $Y$, respectively. Namely, these are defined by

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q),$$
$$\int_0^\infty \mathrm{e}^{-\theta x} \mathbb{W}^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - \delta\theta - q}, \quad \theta > \varphi(q),$$

where

$$\Phi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) = q\},$$
$$\varphi(q) := \sup\{\lambda \geq 0 : \psi(\lambda) - \delta\lambda = q\}.$$

# Fluctuations of Lévy Processes

- Let us define

$$\tau_a^- := \inf\{t > 0 : Y_t < a\} \quad \text{and} \quad \tau_a^+ := \inf\{t > 0 : Y_t > a\}.$$

- Then, for any $a > b$ and $x \leq a$,

$$\mathbb{E}_x\left(e^{-q\tau_a^+}1_{\{\tau_a^+ < \tau_b^-\}}\right) = \frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)},$$

$$\mathbb{E}_x\left(e^{-q\tau_b^-}1_{\{\tau_a^+ > \tau_b^-\}}\right) = \mathbb{Z}^{(q)}(x-b) - \mathbb{Z}^{(q)}(a-b)\frac{\mathbb{W}^{(q)}(x-b)}{\mathbb{W}^{(q)}(a-b)},$$

where

$$\overline{\mathbb{W}}^{(q)}(x) := \int_0^x \mathbb{W}^{(q)}(y)\mathrm{d}y,$$

$$\mathbb{Z}^{(q)}(x) := 1 + q\overline{\mathbb{W}}^{(q)}(x).$$

## Resolvents and Overshoot

- The $q$-*resolvent measure* is

$$\mathbb{E}_x\Big(\int_0^{\tau_b^-\wedge\tau_a^+} e^{-qt}1_{\{Y_t\in \mathrm{d}y\}}\mathrm{d}t\Big) = \Big[\frac{\mathbb{W}^{(q)}(x-b)\mathbb{W}^{(q)}(a-y)}{\mathbb{W}^{(q)}(a-b)} - \mathbb{W}^{(q)}(x-y)\Big]\mathrm{d}y.$$

- It is known that a spectrally negative Lévy process creeps downwards (i.e. $\mathbb{P}_x(Y_{\tau_b^-} = b, \tau_b^- < \infty) > 0$ for $x > b$) iff $\sigma > 0$.
- Hence, *for the case of bounded variation*,

$$\mathbb{E}_x\left(e^{-q\tau_b^-} l(Y_{\tau_b^-})1_{\{\tau_b^- < \tau_a^+\}}\right) = \int_0^{a-b}\int_{(-\infty,-y)} l(y+u+b) \left\{\frac{\mathbb{W}^{(q)}(x-b)\mathbb{W}^{(q)}(a-b-y)}{\mathbb{W}^{(q)}(a-b)} - \mathbb{W}^{(q)}(x-b-y)\right\}\Pi(\mathrm{d}u)\mathrm{d}y.$$

# Fluctuation of Reflected Processes

- Let $\kappa_b^+ := \inf\{t > 0 : U_t > b\}$.
- Its resolvent is

$$\mathbb{E}_x\left(\int_0^{\kappa_b^+} e^{-qt} 1_{\{U_t \in B\}} \mathrm{d}t\right) = \frac{Z^{(q)}(x)}{Z^{(q)}(b)} \int_0^b W^{(q)}(b-y) 1_{\{y \in B\}} \mathrm{d}y - \int_0^x W^{(q)}(x-y) 1_{\{y \in B\}} \mathrm{d}y.$$

- In particular, $\mathbb{E}_x\left(e^{-q\kappa_b^+}\right) = Z^{(q)}(x)/Z^{(q)}(b)$.
- If $\tilde{R}_t := \sup_{s \le t}(-X_t) \vee 0$ so that $U_t = X_t + \tilde{R}_t$,

$$\mathbb{E}_x\left(\int_{[0,\kappa_b^+]} e^{-qt} \mathrm{d}\tilde{R}_t\right) = -\left(\overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q}\right) + \left(\overline{Z}^{(q)}(b) + \frac{\psi'(0+)}{q}\right) \frac{Z^{(q)}(x)}{Z^{(q)}(b)}.$$

# Resolvents

For $q \geq 0$, $x \leq a$ and a Borel set $B$ on $[0, a]$,

$$\mathbb{E}_x\left(\int_0^{T_a^+} e^{-qt} 1_{\{V_t \in B\}} \mathrm{d}t\right) = \int_B \left(w^{(q)}(a, z)\frac{r^{(q)}(x)}{r^{(q)}(a)} - w^{(q)}(x, z)\right)\mathrm{d}z,$$

where

$$\begin{aligned} r^{(q)}(x) &:= Z^{(q)}(x) + q\delta \int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)}(y)\mathrm{d}y, \\ w^{(q)}(x,z) &:= 1_{\{0<z<b\}}\Big(W^{(q)}(x-z) \\ &\qquad + \delta\int_b^x \mathbb{W}^{(q)}(x-y)W^{(q)\prime}(y-z)\mathrm{d}y\Big) \\ &\quad + 1_{\{b<z<a\}}\mathbb{W}^{(q)}(x-z). \end{aligned}$$

## Sketch of Proof 1 (Bounded Var. Case)

For $x < b$, using the strong Markov property and because $T_b^+ = \kappa_b^+$ (hitting time of $U$) and $V_t = U_t$,

$$
\begin{aligned}
f^{(q)}(x, a; B) &= \mathbb{E}_x\Big( \int_0^{\kappa_b^+} e^{-qt} 1_{\{U_t \in B\}} \mathrm{d}t \Big) + \mathbb{E}_x\big(e^{-q\kappa_b^+}\big) f^{(q)}(b, a; B) \\
&= \frac{Z^{(q)}(x)}{Z^{(q)}(b)} \Big[ \int_0^b W^{(q)}(b-y) 1_{\{y \in B\}} \mathrm{d}y + f^{(q)}(b, a; B) \Big] \\
&- \int_0^x W^{(q)}(x-y) 1_{\{y \in B\}} \mathrm{d}y.
\end{aligned}
$$

## Sketch of Proof 2 (Bounded Var. Case)

For $x \geq b$ because $T_b^- = \tau_b^-$ (hitting time of $Y$) and $V_t = Y_t$ on $0 \leq t < T_b^-$ and $V_{T_b^- -} + \Delta X_{T_b^-} = Y_{\tau_b^-}$,

$$
\begin{aligned}
f^{(q)}(x, a; B) &= \mathbb{E}_x\Big( \int_0^{\tau_a^+ \wedge \tau_b^-} e^{-qt} 1_{\{Y_t \in B\}} \mathrm{d}t \Big) \\
&+ \mathbb{E}_x \left( e^{-q\tau_b^-} f^{(q)}(Y_{\tau_b^-}, a; B) 1_{\{\tau_b^- < \tau_a^+\}} \right) \\
&= \int_b^a 1_{\{y \in B\}} \left\{ \frac{\mathbb{W}^{(q)}(x-b)\mathbb{W}^{(q)}(a-y)}{\mathbb{W}^{(q)}(a-b)} - \mathbb{W}^{(q)}(x-y) \right\} \mathrm{d}y \\
&+ \int_0^{a-b} \int_{(-\infty,-y)} f^{(q)}(y+u+b, a; B) \\
&\times \left\{ \frac{\mathbb{W}^{(q)}(x-b)\mathbb{W}^{(q)}(a-b-y)}{\mathbb{W}^{(q)}(a-b)} - \mathbb{W}^{(q)}(x-b-y) \right\} \Pi(\mathrm{d}u)\mathrm{d}y.
\end{aligned}
$$

# Sketch of Proof 3 (Bounded Var. Case)

Solving for $f^{(q)}(b, a; B)$ (+ using some simplification formula), we obtain

$$f^{(q)}(b, a; B) = Z^{(q)}(b) \frac{\int_B w^{(q)}(a, z)\mathrm{d}z}{r^{(q)}(a)} - \int_0^b W^{(q)}(b - y) 1_{\{y \in B\}} \mathrm{d}y.$$

Plugging this back in the previous relations, we obtain $f^{(q)}(x, a; B)$ for all $x \geq 0$.

# Extension to Unbounded Var. Case

1. Given a stochastic process $(\xi_s; s \geq 0)$, a sequence of processes $\{(\xi_s^{(n)})_{s\geq 0}; n \geq 1\}$ is strongly approximating for $\xi$, if $\lim_{n\uparrow\infty} \sup_{0\leq s\leq t} |\xi_s - \xi_n^{(n)}| = 0$ for any $t > 0$ a.s.
2. For any $X$ of unbounded variation, we can construct a strongly approximating sequence $X^{(n)}$ of bounded variation. The corresponding refracted-reflected processes $V^{(n)}$ are strongly approximating for $V$.
3. The corresponding scale functions $W_n^{(q)}$ and $\mathbb{W}_n^{(q)}$ converge to $W^{(q)}$ and $\mathbb{W}^{(q)}$.

– Alternative proof – via excursion theory.

## Resolvents (Cont'd)

(i) For $q > 0$,

$$\mathbb{E}_x\Big(\int_0^\infty e^{-qt} 1_{\{V_t \in B\}} \mathrm{d}t\Big) = \int_B \Big(\frac{e^{-\varphi(q)z} 1_{\{b<z\}} + \delta 1_{\{0<z<b\}} \int_b^\infty e^{-\varphi(q)u} W^{(q)\prime}(u-z)\mathrm{d}u}{\delta q \int_b^\infty e^{-\varphi(q)y} W^{(q)}(y)\mathrm{d}y} r^{(q)}(x) - w^{(q)}(x,z)\Big)\mathrm{d}z.$$

(ii) For $q = 0$ and $\psi_Y'(0+) > 0$ (or $Y_t \xrightarrow{t\uparrow\infty} \infty$ a.s.), then

$$\mathbb{E}_x\Big(\int_0^\infty 1_{\{V_t \in B\}} \mathrm{d}t\Big) = \int_B \Big(\frac{1_{\{b<z\}} + 1_{\{0<z<b\}}\big(1 - \delta^{-1} W(b-z)\big)}{\psi_Y'(0+)} - w^{(0)}(x,z)\Big)\mathrm{d}z.$$

For $q = 0$ and $\psi_Y'(0+) \leq 0$, it becomes infinity given $Leb(B) > 0$.

# Application of Resolvents

For any $q \geq 0$ and $x \leq a$, we have

$$\mathbb{E}_x \left( e^{-qT_a^+} \right) = \frac{r^{(q)}(x)}{r^{(q)}(a)}.$$

In particular, $T_a^+ < \infty$ $\mathbb{P}_x$-a.s.

## Application of Resolvents (Cont'd)

Recall

$$L_t = \delta \int_0^t 1_{\{V_s > b\}} \mathrm{d}s, \quad t \geq 0.$$

- For any $q \geq 0$, $x \leq a$ we have

$$\mathbb{E}_x \left( \int_0^{T_a^+} e^{-qt} \mathrm{d}L_t \right) = \delta \overline{\mathbb{W}}^{(q)}(a-b) \frac{r^{(q)}(x)}{r^{(q)}(a)} - \delta \overline{\mathbb{W}}^{(q)}(x-b).$$

- Fix $x \in \mathbb{R}$. For $q > 0$, we have

$$\mathbb{E}_x \left( \int_0^{\infty} e^{-qt} \mathrm{d}L_t \right) = e^{-\varphi(q)b} \frac{r^{(q)}(x)}{\varphi(q) q \int_b^{\infty} e^{-\varphi(q)y} W^{(q)}(y) \mathrm{d}y} - \delta \overline{\mathbb{W}}^{(q)}(x-b).$$

For $q = 0$, it becomes infinity.

# Capital Injection

Assume $\psi'(0+) > -\infty$ and $q > 0$. For any $x \le a$, we have

$$\mathbb{E}_x\left(\int_{[0,T_a^+]} e^{-qt}\mathrm{d}R_t\right) = \tilde{r}^{(q)}(a)\frac{r^{(q)}(x)}{r^{(q)}(a)} - \tilde{r}^{(q)}(x),$$

$$\mathbb{E}_x\left(\int_{[0,\infty)} e^{-qt}\mathrm{d}R_t\right) = -\tilde{r}^{(q)}(x) + \left(\int_b^\infty e^{-\varphi(q)(y-b)}Z^{(q)}(y)\mathrm{d}y\right) \times \frac{r^{(q)}(x)}{q\int_b^\infty e^{-\varphi(q)(y-b)}W^{(q)}(y)\mathrm{d}y},$$

where

$$\tilde{r}^{(q)}(x) := \overline{Z}^{(q)}(x) + \frac{\psi'(0+)}{q} + \delta\int_b^x \mathbb{W}^{(q)}(x-y)Z^{(q)}(y)\mathrm{d}y, \quad x \in \mathbb{R}.$$

# Occupation Time

For any $p \geq 0$, $q \geq -p$, $a > 0$ and $x \leq a$,

$$\mathbb{E}_x\left(e^{-pT_a^+ - q\int_0^{T_a^+} 1_{\{V_s<b\}}\mathrm{d}s}\right) = \frac{\mathcal{R}^{(p,q)}(x)}{\mathcal{R}^{(p,q)}(a)},$$

$$\mathbb{E}_x\left(e^{-pT_a^+ - q\int_0^{T_a^+} 1_{\{V_s>b\}}\mathrm{d}s}\right) = \frac{\mathcal{L}^{(p,q)}(x)}{\mathcal{L}^{(p,q)}(a)},$$

where

$$\begin{aligned}\mathcal{R}^{(p,q)}(x) &:= Z^{(p+q)}(x) - q\overline{\mathbb{W}}^{(p)}(x-b) \\ &\quad - (p+q)\int_b^x \mathbb{W}^{(p)}(x-y)\left(q\overline{W}^{(p+q)}(y) - \delta W^{(p+q)}(y)\right)\mathrm{d}y, \\ \mathcal{L}^{(p,q)}(x) &= Z^{(p)}(x) + q\overline{\mathbb{W}}^{(p+q)}(x-b) \\ &\quad + p\int_b^x \mathbb{W}^{(p+q)}(x-y)\left(q\overline{W}^{(p)}(y) + \delta W^{(p)}(y)\right)\mathrm{d}y.\end{aligned}$$

# Optimization w/ singular&abs. cont. control

- Joint work with B. Avanzi, B. Wong, and J.L. Pérez.
- $Y$ is a spectrally positive Lévy process.
- A dividend strategy $\pi := (A^\pi_t, S^\pi_t; t \geq 0)$
  - $S^\pi$: usual control (nondecreasing, right-continuous, and adapted)
  - $A^\pi$: absolutely continuous control $A^\pi_t = \int_0^t a^\pi_s \mathrm{d}s$, $t \geq 0$, with $a^\pi$ restricted to take values in $[0, \delta]$ uniformly in time.
- The controlled risk process becomes

$$U^\pi_t := Y_t - A^\pi_t - S^\pi_t, \quad t \geq 0.$$

# Optimization w/ singular&abs. cont. control

- We want to maximize

$$v_\pi(x) = \mathbb{E}_x \left( \int_0^{\sigma^\pi} e^{-qt} \mathrm{d}A_t^\pi + \beta \int_{[0,\sigma^\pi]} e^{-qt} \mathrm{d}S_t^\pi + \rho e^{-q\sigma^\pi} \right),$$

where

$$\sigma^\pi := \inf\{t > 0 : U_t^\pi < 0\},$$

is the time to ruin.

- To activate $S^\pi$, one needs to pay (proportional) costs:$S^\pi$:

$$0 < \beta < 1.$$

- $\rho \in \mathbb{R}$ is a terminal reward/penalty.
- Let $\mathcal{A}$ be the set of all admissible strategies that satisfy the above conditions and

$$\Delta S_t^\pi \leq U_{t-}^\pi + \Delta Y_t, \quad t \geq 0.$$

# Two layer $(a, b)$-strategy

- It is conjectured that it is optimal to
  - activate absolutely continuous control $A$ when the process is above $a$.
  - activate singular control $S$ when the process is above $b > a$.
- Under two-layer $(a, b)$-strategy, the controlled process becomes the refracted-reflected process – flipped case of the ones discussed.
- Let

$$v_{a,b}(x) = \mathbb{E}_x \left( \int_0^{\sigma_{a,b}} e^{-qt} \mathrm{d}A_t^{a,b} + \beta \int_{[0,\sigma_{a,b}]} e^{-qt} \mathrm{d}S_t^{a,b} + \rho e^{-q\sigma_{a,b}} \right),$$

with its ruin time

$$\sigma_{a,b} := \sigma^{\pi_{a,b}} = \inf\{t > 0 : U_t^{a,b} := Y_t - A_t^{a,b} - S_t^{a,b} < 0\}.$$

# NPV under two layer $(a, b)$-strategy

For all $0 \leq a < b$ and $x \geq 0$, we have

$$v_{a,b}(x) = -\frac{\Gamma(a,b)}{q}\frac{r^{(q)}_{b-a}(b-x)}{r^{(q)}_{b-a}(b)} + \frac{\delta}{q}\mathbb{Z}^{(q)}(a-x) - \beta\tilde{r}^{(q)}_{b-a}(b-x),$$

where we define, for $0 \leq a \leq b$,

$$\Gamma(a,b) := \delta\mathbb{Z}^{(q)}(a) - q\rho - q\beta\tilde{r}^{(q)}_{b-a}(b),$$

$$r^{(q)}_{b-a}(z) := Z^{(q)}(z) + q\delta\int_{b-a}^{z}\mathbb{W}^{(q)}(z-y)W^{(q)}(y)\mathrm{d}y,$$

$$\tilde{r}^{(q)}_{b-a}(z) := \overline{Z}^{(q)}(z) + \frac{\psi'(0+)}{q} + \delta\int_{b-a}^{z}\mathbb{W}^{(q)}(z-y)Z^{(q)}(y)\mathrm{d}y.$$

# Selection of $(a^*, b^*)$

1. If condition
$$\mathbf{C}_b : \Gamma(a, b) = 0$$
holds, then $v_{a,b}$ is continuously differentiable (resp. twice continuously differentiable) at $b$ when $Y$ has paths of bounded (resp. unbounded) variation.
2. Additionally if condition
$$\mathbf{C}'_a : \gamma(a, b) := \beta^{-1} - Z^{(q)}(b - a) = 0$$
holds, then $v_{a,b}$ is continuously differentiable (resp. twice continuously differentiable) at $a$ when $Y$ has paths of bounded (resp. unbounded) variation.

# Existence of $(a^*, b^*)$

- We have $\frac{\partial}{\partial a}\Gamma(a,b) = \delta q \beta \mathbb{W}^{(q)}(a)\gamma(a,b)$.
- If $a \mapsto \Gamma(a, b^*)$ gets tangent to the $x$-axis at $a^* > 0$, then necessarily $\Gamma(a^*, b^*) = \gamma(a^*, b^*) = 0$.

There exist a pair $(a^*, b^*)$ such that one of the following holds.

(i) $a^* = b^* = 0$ w/ $\Gamma(0) = \delta - q\rho - \beta\psi_X'(0+) \leq 0$.

(ii-1) $a^* = 0 < b^*$ w/ $\Gamma(a^*, b^*) = 0$ and $\beta^{-1} - Z^{(q)}(b^*) \geq 0$, and $\Gamma(0) > 0$.

(ii-2) $0 < a^* < b^*$ w/ $\Gamma(a^*, b^*) = \gamma(a^*, b^*) = 0$, and $\Gamma(0) > 0$.

## Form of $v_{a^*,b^*}$

- For $b^* > 0$,

$$v_{a^*,b^*}(x) = \frac{\delta}{q}\mathbb{Z}^{(q)}(a^* - x) - \beta \tilde{r}^{(q)}_{b^*-a^*}(b^* - x), \quad x \geq 0,$$

  where in particular

$$v_{a^*,b^*}(x) = \frac{\delta}{q} - \beta\Big(\overline{Z}^{(q)}(b^* - x) + \frac{\psi'(0+)}{q}\Big), \quad a^* \leq x,$$

$$v_{a^*,b^*}(x) = \beta\Big(x - b^* - \frac{\psi'_X(0+)}{q}\Big) + \frac{\delta}{q}, \quad x \geq b^*.$$

- On the other hand, for $a^* = b^* = 0$,

$$v_{0,0}(x) = \beta x + \rho, \quad x \geq 0.$$

- For both cases, it can be confirmed that

$$v_{a^*,b^*}(0) = \lim_{x \downarrow 0} v_{a^*,b^*}(x) = \rho.$$

# Verification lemma

Suppose $\hat{\pi}$ is an admissible dividend strategy such that

1. $v_{\hat{\pi}}$ is sufficiently smooth ($C^1(0,\infty)$ [resp. $C^2(0,\infty)$] when $X$ has paths of bounded [resp. unbounded] variation) on $(0,\infty)$,
2. it satisfies

$$\sup_{0\le r\le\delta}\left((\mathcal{L}_Y - q)v_{\hat{\pi}}(x) - r v_{\hat{\pi}}'(x) + r\right) \le 0, \quad x > 0,$$
$$v_{\hat{\pi}}'(x) \ge \beta, \quad x > 0.$$

3. $\rho = v_{\hat{\pi}}(0) \le \lim_{x\downarrow 0} v_{\hat{\pi}}(x)$.

Then $v_{\hat{\pi}}(x) = v(x)$ for all $x \ge 0$ and hence $\hat{\pi}$ is an optimal strategy.

# Main results

Lemma

*The function* $v_{a^*,b^*}$ *is concave and the following holds:*

1. *For* $x > a^*$, *we have* $\beta \le v'_{a^*,b^*}(x) \le 1$;
2. *For* $0 < x < a^*$, *we have* $v'_{a^*,b^*}(x) \ge 1 > \beta$.
3. *Suppose* $a^* > 0$. *For* $0 < x < a^*$, *we have* $(\mathcal{L}_Y - q)v_{a^*,b^*}(x) = 0$.
4. *Suppose* $b^* > 0$. *For* $a^* < x < b^*$, *we have* $(\mathcal{L}_X - q)v_{a^*,b^*}(x) + \delta = 0$.

(iii) *For* $x > b^*$, $(\mathcal{L}_X - q)v_{a^*,b^*}(x) + \delta \le 0$.

Theorem

*The two-layer* $(a^*, b^*)$ *strategy for* $(a^*, b^*)$ *is optimal, and the value function is given by* $v(x) = v_{a^*,b^*}(x)$ *for all* $0 \le x < \infty$.

## Numerical results

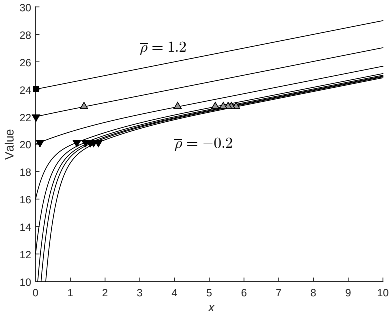


Figure: Sensitivity of the value function $v(x)$ with respect to $\bar{\rho} := q\rho/\delta$.

# Numerical results

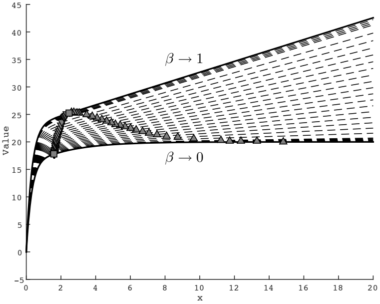


Figure: Sensitivity of the value function $v(x)$ with respect to $\beta = [0.01, 0.02, 0.03, 0.04, 0.05, 0.1, ..., 0.90, 0.95, 0.96, 0.97, 0.98, 0.99]$.

# Numerical results

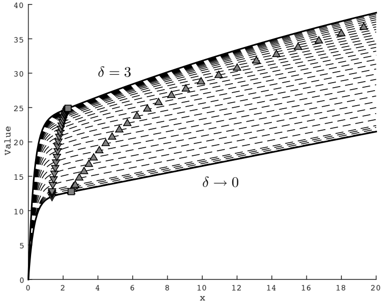


Figure: Sensitivity of the value function $v(x)$ with respect to $\delta = [0.01, 0.04, 0.07, 0.1, 0.2, \ldots, 2.9, 3]$.

# References


[1] J. L. Pérez, K. Yamazaki. *On the Refracted-Reflected Spectrally Negative Lvy Processes.* arXiv:1511.06027, 2015.

[2] J. L. Pérez, K. Yamazaki. *Refraction-reflection strategies in the dual model.* arXiv:1511.07918, 2015.

[3] B. Avanzi, J. L. Pérez, B. Wong, K. Yamazaki. *On the Joint Reflective and Refractive Dividend Strategies in Spectrally Positive Lévy Processes.* arXiv:1607.01902, 2016.