Generalized refracted Lévy process and its application to exit problem

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Exit problem for a spectrally negative process:

$$\mathbb{E}_x^Z\left[e^{-q\tau_a^+}; \tau_a^+ < \tau_b^-\right] = \frac{W_Z^{(q)}(x-b)}{W_Z^{(q)}(a-b)} \quad (b < x < a) \tag{1}$$

Kyprianou–Loeffen (2010), exit problem for a refracted Lévy process:

$$\mathrm{d}U_t = \mathrm{d}X_t + \alpha 1_{\{U_{t-}<0\}}\mathrm{d}t \tag{2}$$

for X being a spectrally negative (non-monotonic) Lévy process

$$\mathbb{E}_x^U\left[e^{-q\tau_a^+}; \tau_a^+ < \tau_b^-\right] = \frac{W_U^{(q)}(x,b)}{W_U^{(q)}(a,b)} \quad \begin{pmatrix} b < 0 < a \\ b < x < a \end{pmatrix} \tag{3}$$

Generalize Kyprianou–Loeffen's results to a process such that

$$\begin{cases} \text{If } U_0 > 0,\ (U_t)_{t<\tau_0^-} \overset{\text{law}}{=} (X_t)_{t<\tau_0^-} \\ \text{If } U_0 < 0,\ (U_t)_{t<\tau_0^+} \overset{\text{law}}{=} (Y_t)_{t<\tau_0^+} \end{cases} \tag{4}$$

for two spec. neg. X and Y where $Y - X$ is not a positive drift.

1 Construction

Kyprianou–Loeffen's SDE:

$$\mathrm{d}U_t = \mathrm{d}X_t + \alpha 1_{\{U_{t-}<0\}}\mathrm{d}t = \begin{cases} \mathrm{d}X_t & (U_{t-} \geq 0) \\ \mathrm{d}(X_t + \alpha t) & (U_{t-} < 0) \end{cases} \tag{5}$$

One may expect to generalize it by the following SDE:

$$\mathrm{d}U_t = 1_{\{U_{t-}\geq 0\}}\mathrm{d}X_t + 1_{\{U_{t-}<0\}}\mathrm{d}Y_t = \begin{cases} \mathrm{d}X_t & (U_{t-} \geq 0) \\ \mathrm{d}Y_t & (U_{t-} < 0) \end{cases} \tag{6}$$

where X and Y are independent. In the special case $Y_t \stackrel{\text{law}}{=} X_t + \alpha t$, although the SDE (5) is apparently different from (6), the solutions of them are equivalent in law.

$$\mathrm{d}U_t = 1_{\{U_{t-}\geq 0\}}\mathrm{d}X_t + 1_{\{U_{t-}<0\}}\mathrm{d}Y_t = \begin{cases} \mathrm{d}X_t & (U_{t-} \geq 0) \\ \mathrm{d}Y_t & (U_{t-} < 0) \end{cases} \tag{6}$$

In the case X is of bounded variation, we can construct a solution to (6) in the same way as Kyprianou–Loeffen. This is as follows: If we have constructed U_t up to the $(n-1)$th time of zero for U, which we call T_{n-1}, then U_t for $T_{n-1} < t \leq T_n$ is defined as

$$\begin{aligned} U_t =& X_t - X_{T_{n-1}} \quad (T_{n-1} < t \leq T'_n := \inf\{t > T_{n-1} : U_t < 0\}) \\ U_t =& U_{T'_n} + Y_t - Y_{T'_n} \quad (T'_n < t \leq T_n := \inf\{t > T'_n : U_t = 0\}) \end{aligned}$$

In the general case, however, we do not know how to prove existence of a solution to (6)...

Recall the proof of the **uniqueness for the Kyprianou–Loeffen's SDE**:

$$\mathrm{d}U_t = \mathrm{d}X_t + \alpha 1_{\{U_{t-}<0\}}\mathrm{d}t, \qquad (2)$$

Let $U^{(i)}$, $i = 1, 2$ be two solutions with a common driving noise X.

Then the difference $\Delta_t := U_t^{(1)} - U_t^{(2)}$ satisfies

$$\Delta_t^2 = 2\alpha \int_0^t \Delta_s \left(1_{\{U_{s-}^{(1)}<0\}} - 1_{\{U_{s-}^{(2)}<0\}} \right) \mathrm{d}s \leq 0. \qquad (7)$$

For the **general SDE**:

$$\mathrm{d}U_t = 1_{\{U_{t-}\geq 0\}}\mathrm{d}X_t + 1_{\{U_{t-}<0\}}\mathrm{d}Y_t = \begin{cases} \mathrm{d}X_t & (U_{t-} \geq 0) \\ \mathrm{d}Y_t & (U_{t-} < 0) \end{cases} \qquad (6)$$

the uniqueness can be easily obtained **if X and Y are both compound Poisson with drifts**. **In the general case, however, we do not know how to prove existence nor uniqueness for (6)...**

We discard the SDE approach and appeal to the excursion theory.

Assume: X and Y are spectrally negative Lévy processes of unbounded variation and X has no Gaussian component

n^X: the excursion measure away from zero with normalization:

$$n^X\left[1-\mathrm{e}^{-qT_0}\right]=\frac{1}{r_X^{(q)}(0,0)}=\Psi_X'(\Psi_X^{-1}(q)) \tag{8}$$

where $\Psi_X(q)=\log\mathbb{E}_0^X[\mathrm{e}^{qX_1}]$. This identity is equivalent to

$$\mathbb{E}_0^X\left[\int_0^\infty \mathrm{e}^{-qt}\mathrm{d}L_t\right]=r_X^{(q)}(0,0), \tag{9}$$

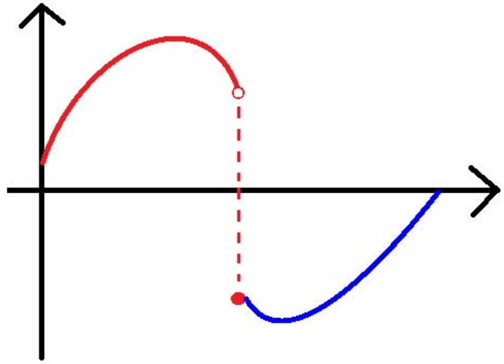
which shows that the local time L_t at zero is chosen via Revuz's correspondence between L_t and the Dirac delta at zero.

Define n^U and $\mathbb{P}_x^{U^0}$ as follows:

$$n^U\left[F\left((U_t)_{t<\tau_0^-}, \theta_{\tau_0^-}U\right)\right] = n^X\left[\mathbb{E}_y^{Y^0}[F(w, Y^0)]\Big|_{\substack{y=X(\tau_0^-)\\ w=(X_t)_{t<\tau_0^-}}}\right] \quad (10)$$

$$\mathbb{P}_x^{U^0}\left[F\left((U_t^0)_{t<\tau_0^-}, \theta_{\tau_0^-}U^0\right)\right] = \mathbb{P}_x^X\left[\mathbb{E}_y^{Y^0}[F(w, Y^0)]\Big|_{\substack{y=X(\tau_0^-)\\ w=(X_t)_{t<\tau_0^-}}}\right] \quad (11)$$

Thm 1 U is a Feller process.



Thm 2 n^U satisfies

$$n^U\left[1-\mathrm{e}^{-qT_0}\right] = \frac{1}{r_U^{(q)}(0, 0+)} = \frac{1}{\lim_{y\downarrow 0} r_U^{(q)}(0, y)}. \quad (12)$$

Rem 3 If Y has Gaussian component, then we may define

$$n^U(\cdot) = c^+ n^U(\cdot;\Omega^+) + c^- n^U(\cdot;\Omega^-) \tag{13}$$

$$\Omega^+ = \{U_t > 0 \text{ for any small } t > 0\} \tag{14}$$

$$\Omega^- = \{U_t < 0 \text{ for any small } t > 0\} \tag{15}$$

$$n^U\left[F\left((U_t)_{t<\tau_0^-}, \theta_{\tau_0^-}U\right);\Omega^+\right] = n^X\left[\mathbb{E}_y^{Y^0}\left[F(w, Y^0)\right]\Big|_{\substack{y=X(\tau_0^-)\\ w=(X_t)_{t<\tau_0^-}}}\right] \tag{16}$$

$$n^U\left[F(U);\Omega^-\right] = n^Y[F(Y)] \tag{17}$$

Note that under n^Y the process stays negative until it hits zero and then stops at zero.

We will see later why we ignore excursions in Ω^-.

2 Exit problems

Exit problem for Kyprianou–Loeffen's refracted Lévy process:

$$\mathbb{E}_x^U\left[\mathrm{e}^{-q\tau_a^+};\tau_a^+<\tau_b^-\right]=\frac{W_U^{(q)}(x,b)}{W_U^{(q)}(a,b)}\qquad\begin{pmatrix}b<0<a\\ b<x<a\end{pmatrix}\tag{3}$$

where

$$W_U^{(q)}(x,y)=W_Y^{(q)}(x-y)\qquad (x\le 0)$$

$$W_U^{(q)}(x,y)=W_Y^{(q)}(x-y)+\alpha\int_0^x W_X^{(q)}(x-z)W_Y^{(q)\prime}(z-y)\mathrm{d}z\qquad (x>0)$$

Thm 4 **Exit problem for our generalized refracted Lévy process:**

$$\mathbb{E}_x^U\left[e^{-q\tau_a^+};\tau_a^+<\tau_b^-\right]=\frac{W_U^{(q)}(x,b)}{W_U^{(q)}(a,b)}\quad\begin{pmatrix}b<0<a\\ b<x<a\end{pmatrix}\tag{3}$$

where

$$W_U^{(q)}(x,y)=W_Y^{(q)}(x-y)\qquad (x\le 0)$$

$$W_U^{(q)}(x,y)=H_1^{(q)}(x,y)+\int H_2^{(q)}(x,y;u,v)\widetilde{\Pi}_X(\mathrm{d}u\mathrm{d}v)\qquad (x>0)$$

$$H_1^{(q)}(x,y)=W_X^{(q)}(x)W_Y^{(q)}(-y)(\Psi'_X(0)\vee 0)$$

$$H_2^{(q)}(x,y;u,v)=W_X^{(q)}(x)W_Y^{(q)}(-y)e^{\Phi_Y(0)u}-W_Y^{(q)}(u-y)W_X^{(q)}(x-v)$$

$$\widetilde{\Pi}_X(\mathrm{d}u\mathrm{d}v)=\Pi_X(\mathrm{d}u-v)\mathrm{d}v\quad \text{on } (-\infty,0)\times(0,\infty).$$

Rem 5 Comparison of $W_U^{(q)}(x,y)$ for $x > 0$

Kyprianou–Loeffen's formula:

$$W_U^{(q)}(x,y) = W_Y^{(q)}(x-y) + \alpha \int_0^x W_X^{(q)}(x-z) W_Y^{(q)\prime}(z-y) \mathrm{d}z$$

Our formula:

$$W_U^{(q)}(x,y) = H_1^{(q)}(x,y) + \int H_2^{(q)}(x,y;u,v) \widetilde{\Pi}_X(\mathrm{d}u\mathrm{d}v)$$

$$\begin{aligned} H_1^{(q)}(x,y) =& W_X^{(q)}(x) W_Y^{(q)}(-y) (\Psi'_X(0) \vee 0) \\ H_2^{(q)}(x,y;u,v) =& W_X^{(q)}(x) W_Y^{(q)}(-y) \mathrm{e}^{\Phi_Y(0)u} - W_Y^{(q)}(u-y) W_X^{(q)}(x-v) \\ \widetilde{\Pi}_X(\mathrm{d}u\mathrm{d}v) =& \Pi_X(\mathrm{d}u - v)\mathrm{d}v \quad \text{on } (-\infty,0) \times (0,\infty). \end{aligned}$$

Note that our formula involves the Lévy measure Π_X, while Kyprianou–Loeffen made some special efforts so that their formula does not involve the Lévy measure Π_X explicitly.

Potential measures with absorbing barriers

$$\underline{\overline{R}}_U^{(q;b,a)} f(x) = \mathbb{E}_x^U \left[\int_0^{\tau_a^+ \wedge \tau_b^-} \mathrm{e}^{-qt} f(U_t) \mathrm{d}t \right] \tag{18}$$

Thm 6 Density representation:

$$\underline{\overline{r}}_U^{(q;b,a)}(x,y) = \frac{W_U^{(q)}(x,b)}{W_U^{(q)}(a,b)} W_X^{(q)}(a-y) - W_X^{(q)}(x-y) \qquad (y \in (0,a]) \tag{19}$$

$$\underline{\overline{r}}_U^{(q;b,a)}(x,y) = \frac{W_U^{(q)}(x,b)}{W_U^{(q)}(a,b)} W_U^{(q)}(a,y) - W_U^{(q)}(x,y) \qquad (y \in [b,0]) \tag{20}$$

Note that these formulae are of the same form as Kyprianou–Loeffen's.

3 Approximation

For a general spectrally negative (non-monotone) Lévy process Z with Laplace exponent

$$\Psi_Z(q) = \gamma_Z q + \frac{\sigma_Z^2}{2} q^2 - \int_{(-\infty,0)} \left(1 - e^{qy} + qy1_{(-1,0)}(y)\right) \Pi_Z(dy) \tag{21}$$

we define $Z^{(n)}$ as a compound Poisson with positive drift:

$$\Psi_{Z^{(n)}}(q) = \delta_{Z^{(n)}} q - \int_{(-\infty,0)} \left(1 - e^{qy}\right) \Pi_{Z^{(n)}}(dy) \tag{22}$$

where

$$\delta_{Z^{(n)}} q = \gamma_Z + \sigma_Z^2 n + \int_{(-1,-1/n)} (-y) \Pi_Z(dy) \tag{23}$$

$$\Pi_{Z^{(n)}} = 1_{(-\infty,-1/n)} \Pi_Z + \sigma_Z^2 n^2 \delta_{(-1/n)} \tag{24}$$

It is obvious that $Z^{(n)} \xrightarrow{\mathbb{D}} Z$ in law. Moreover, it is known that we can find a coupling such that $Z^{(n)} \xrightarrow{\text{u.c.}} Z$ a.s.

Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be spectrally negative and assume that $\boldsymbol{X}$ has no Gaussian component. Let U denote our generalized refracted Lévy process. For each n, let $\boldsymbol{X}^{(n)}$ and $\boldsymbol{Y}^{(n)}$ be realized on a common probability space such that they are independent. Let $U^{(n)}$ be the unique solution to the SDE:

$$\mathrm{d}U_t^{(n)} = 1_{\{U_{t-}^{(n)} \geq 0\}} \mathrm{d}\boldsymbol{X}_t^{(n)} + 1_{\{U_{t-}^{(n)} < 0\}} \mathrm{d}\boldsymbol{Y}_t^{(n)} = \begin{cases} \mathrm{d}\boldsymbol{X}_t^{(n)} & (U_{t-}^{(n)} \geq 0) \\ \mathrm{d}\boldsymbol{Y}_t^{(n)} & (U_{t-}^{(n)} < 0) \end{cases} \tag{6}$$

Thm 7 $U^{(n)} \xrightarrow{\mathbb{D}} U$ in law.

(This is why we ignored excursions with negative germs.)

(We do not know whether we can find a u.c. a.s. coupling.)

4 Sketch of the proofs

Recall the Gerber–Shiu formula:

$$\mathbb{E}_x^X\left[\mathrm{e}^{-q\tau_0^-} f(X_{\tau_0^-}, X_{\tau_0^- -})\right] = \int f(u,v) G_X^{(q)}(x, \mathrm{d}u\mathrm{d}v) \tag{25}$$

$$G_X^{(q)}(x, \mathrm{d}u\mathrm{d}v) = \underline{r}_X^{(q;0)}(x,v) \widetilde{\Pi}_X(\mathrm{d}u\mathrm{d}v) \tag{26}$$

$$\underline{r}_X^{(q;0)}(x,v) = \mathrm{e}^{-\Phi_X(q)y} W_X^{(q)}(x) - W_X^{(q)}(x-y) \tag{27}$$

Thm 8 The Gerber–Shiu formula for the excursion measure:

$$n^X\left[\mathrm{e}^{-q\tau_0^-} f(X_{\tau_0^-}, X_{\tau_0^- -})\right] = \int f(u,v) K_X^{(q)}(\mathrm{d}u\mathrm{d}v) \tag{28}$$

$$K_X^{(q)}(\mathrm{d}u\mathrm{d}v) = \mathrm{e}^{-\Phi_X(q)v} \widetilde{\Pi}_X(\mathrm{d}u\mathrm{d}v) \tag{29}$$

Recall the normalization: $n^{X}\left[1-\mathrm{e}^{-qT_0}\right]=\dfrac{1}{r_X^{(q)}(0,0)}$. **(8)**

Lem 9 Denote $\overline{X}=\sup_{t\geq 0} X_t$.

Then $n^{X}\left[\mathrm{e}^{-q\tau_a^+};\overline{X}>a\right]=\dfrac{1}{W_X^{(q)}(a)}$. **(30)**

Consequently, letting $q=0$, we have $n^{X}\left(\overline{X}>a\right)=\dfrac{1}{W_X^{(0)}(a)}$. **(31)**

Rem 10 For general one-dimensional diffusions, the relation between (8) and (31) was obtained in 2015 by Chen–Fukushima and Y–Yano.

Rem 11 Pardo–Pérez–Rivero (2015, arXiv:1507.05225) study a close relation between n^X, the excursion measure of X itself, and the excursion measure of the reflected process of X.

Thm 12 **Resolvents may be represented as follows:**

$$R_U^{(q)} f(0) = \frac{N_U^{(q)} f}{q N_U^{(q)} 1} \qquad (x = 0)$$

$$R_U^{(q)} f(x) = R_{Y^0}^{(q)} f(x) + e^{\Phi_Y(q)x} R_U^{(q)} f(0) \qquad (x < 0)$$

$$R_U^{(q)} f(x) = \underline{R}_X^{(q;0)} f(x) + \int R_U^{(q)} f(u) G_X^{(q)}(x, \mathrm{d}u\mathrm{d}v) \qquad (x > 0)$$

where

$$N_U^{(q)} f := n^U \left[\int_0^{T_0} e^{-qt} f(X_t) \mathrm{d}t \right]$$

$$= \int_0^{\infty} e^{-\Phi_X(q)y} f(y) \mathrm{d}y + \int R_{Y^0}^{(q)} f(u) K_X^{(q)}(\mathrm{d}u\mathrm{d}v)$$

$$\underline{R}_X^{(q;0)} f(x) = \mathbb{E}_x^X \left[\int_0^{\tau_0^-} e^{-qt} f(X_t) \mathrm{d}t \right]$$

Thm 7 $U^{(n)} \xrightarrow{\mathbb{D}} U$ in law.

It is known that for Feller processes convergence in law on $\mathbb{D}$ is equivalent to strong convergence of the corresponding semigroups and to that of the corresponding reslovents. Now Thm 7 reduces to

Thm 13 $R^{(q)}_{U^{(n)}} \to R^{(q)}_U$ strongly on C_0.

The proof is divided into the following steps:

1. pointwise convergence.
2. $R^{(q)}_{U^{(n)}} f$'s vanish uniformly outside some compact interval.
3. $R^{(q)}_{U^{(n)}} f$'s are equicontinuous on any compact interval.