Perpetual Integrals for Lévy Processes

A 0-1 law

Leif Döring

Uni Mannheim

Theorem (D., Kyp '15)

Let $f : \mathbb{R} \to [0, \infty)$ measurable and locally integrable (e.g. continuous) and ξ a Lévy process with

(1) *positive finite mean,*

(2) *existence of local times.*

Then

$$\mathbb{P}\left(\int_0^\infty f(\xi_s)\, ds < \infty\right) \in \{0, 1\}$$

and

$$\mathbb{P}\left(\int_0^\infty f(\xi_s)\, ds < \infty\right) = 1 \quad \Leftrightarrow \quad \int f(x)\, dx < \infty.$$

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- $\int_0^\infty f(\xi_s)\,ds$ is called a perpetual integral
- perpetual integral needed for entrance boundary of jump diffusions
- (1) implies ξ drifts to $+\infty$ a.s.
- (2) means $\int_0^t f(\xi_s)\,ds = \int_{\mathbb{R}} f(x) L_t(x)\,dx$ a.s.
- (1), (2) not necessary!

Examples

- Brownian motion with positive drift

$$\xi_t = \sigma B_t + b\,t, \quad b > 0.$$

  Perpetual integrals studied before by Salminen/Yor.
- Spectrally negative Lévy processes studied by Schilling/Vondraček and Khoshnevisan/Salminen/Yor.

Agenda

- 0-1 law, Hewitt-Savage
- "$\Leftarrow$" potential theory
- "$\Rightarrow$" Jeulin Lemma and fluctuation theory
- Why?

0-1 law, Hewitt-Savage

Define increment processes

$$\xi^n_t = \xi_{n+t} - \xi_n, \quad t \in [0,1],$$

so $\xi_0, \xi_1, \ldots$ is iid. Write

$$\begin{aligned}
\Lambda &= \left\{ \int_0^\infty f(\xi_s)\, ds < \infty \right\} \\
&= \left\{ \sum_{k=0}^\infty \int_k^{k+1} f(\xi_s)\, ds < \infty \right\} \\
&= \left\{ \sum_{k=0}^\infty \int_0^1 f\left(\xi^k_s + \sum_{n=0}^{k-1} \xi^k_1 \right) ds < \infty \right\} \\
&\in \sigma(\xi^0, \xi^1, \ldots).
\end{aligned}$$

Λ is exchangeable (i.e. does not depend on finite permutations), so Hewitt-Savage implies $\mathbb{P}(\Lambda) \in \{0,1\}$.

"$\Leftarrow$", potential theory

Show stronger statement $\mathbb{E}\left[\int_0^\infty f(\xi_s)\, ds\right] < \infty$.

$$\mathbb{E}\left[\int_0^\infty f(\xi_s)\, ds\right] = \int_{\mathbb{R}} f(x) U(dx),$$

where U is the potential measure $U(dx) = \mathbb{E}\left[\int_0^\infty 1_{\xi_s \in dx}\, ds\right]$.

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Potential theory for Lévy processes says, under assumption of the theorem (local time and transience), U has bounded density. Hence,

$$\mathbb{E}\left[\int_0^\infty f(\xi_s)\,ds\right] = \int_{\mathbb{R}} f(x)U(dx) = \int_{\mathbb{R}} f(x)\,u(x)\,dx \le C\int_{\mathbb{R}} f(x)\,dx < \infty$$

by assumption.

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Note: $\int f(x)\,U(dx) < \infty$ is the right condition!

“$\Rightarrow$”, Jeulin Lemma

Lemma (A version of Jeulin's lemma)

Suppose $(X_x)_{x\in\mathbb{R}}$ *are*

- *identically distributed on some probability space* $(\Omega, \mathcal{A}, P)$,
- *non-trivial and non-negative,*
- *(some measurability).*

Then

$$P\left(\int_{\mathbb{R}} f(x)\, X_x\, dx < \infty\right) \quad \Rightarrow \quad \int_{\mathbb{R}} f(x)\, dx < \infty.$$

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