Stochastic integration w.r.t. cylindrical Lévy processes

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Stochastic heat equation

Let $\mathscr{O} \subseteq \mathbb{R}$

$$\frac{\partial X}{\partial t}(t,r) = \frac{\partial^2 X}{\partial r^2}(t,r) \qquad \text{for all } r \in \mathscr{O},\ t \in [0,T].$$

Stochastic heat equation

Let $\mathscr{O} \subseteq \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$:

$$\frac{\partial X}{\partial t}(t,r) = \frac{\partial^2 X}{\partial r^2}(t,r) + f(X(t,r))\underbrace{\frac{\partial N}{\partial t}(t,r)}_{\text{noise in } t \text{ and } r} \qquad \text{for all } r \in \mathscr{O},\ t \in [0,T].$$

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Search solution $X := (X(t,\cdot) : t \in [0,T])$ in $L^2(\mathscr{O})$:

$$\frac{dX}{dt}(t,\cdot) = AX(t,\cdot) + f(X(t,\cdot))\underbrace{\frac{dN}{dt}(t,\cdot)}_{\text{noise in } L^2(\mathscr{O})} \qquad \text{for all } t \in [0,T],$$

where $A : \text{dom}(A) \subseteq L^2(\mathscr{O}) \to L^2(\mathscr{O})$ with $Af = \frac{d^2 f}{dr^2}$.

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where $A : \text{dom}(A) \subseteq L^2(\mathscr{O}) \to L^2(\mathscr{O})$ with $Af = \frac{d^2 f}{dr^2}$. As integral equation:

$$X(t,\cdot) = X(0,\cdot) + \int_0^t AX(s,\cdot)ds + \underbrace{\int_0^t f(X(s,\cdot))\,N(ds,\cdot)}_{\text{stochastic integral}} \qquad \text{for all } t \in [0,T].$$

Cylindrical processes

Cylindrical processes

Let U be a Banach space with dual space U^* and dual pairing $\langle \cdot, \cdot \rangle$ and let $(\Omega, \mathscr{A}, P)$ denote a probability space.

Definition: A cylindrical random variable X in U is a mapping

$$X : U^* \to L_P^0(\Omega; \mathbb{R}) \quad \text{linear and continuous.}$$

A cylindrical process in U is a family $(X(t) : \ t \geqslant 0)$ of cylindrical random variables.

- I. E. Segal, 1954
- I. M. Gel'fand 1956: Generalized Functions
- L. Schwartz 1969: seminaire rouge, radonifying operators

Example: induced cylindrical random variable

Example: Let $X : \Omega \to U$ be a (classical) random variable. Then

$$Z : U^* \to L_P^0(\Omega; \mathbb{R}), \qquad Za := \langle X, a \rangle$$

defines a cylindrical random variable.

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defines a cylindrical random variable.

But: not for every cylindrical random variable $Z : U^* \to L^0_P(\Omega; \mathbb{R})$ there exists a classical random variable $X : \Omega \to U$ satisfying

$$Za = \langle X, a \rangle \qquad \text{for all } a \in U^*.$$

Example: cylindrical Wiener process

Definition:

A cylindrical process $(W(t)\ :\ \ t \geqslant 0)$ is called a *cylindrical Wiener process*, if for all $a_1, \dots, a_n \in U^*$ and $n \in \mathbb{N}$ the stochastic process :

$$\Big((W(t)a_1, \dots, W(t)a_n) :\ t \geqslant 0\Big)$$

is a centralised Wiener process in $\mathbb{R}^n$.

Cylindrical Lévy processes

Definition: cylindrical Lévy process

Definition: (Applebaum, Riedle (2010))
A cylindrical process $(L(t):\ t \geqslant 0)$ is called a *cylindrical Lévy process*, if for all $a_1, \dots, a_n \in U^*$ and $n \in \mathbb{N}$ the stochastic process :

$$\Big((L(t)a_1, \dots, L(t)a_n):\ t \geqslant 0 \Big)$$

is a Lévy process in $\mathbb{R}^n$.

No semimartingale decomposition in U

If $a \in U^*$ then $(L(t)a : t \geqslant 0)$ is a Lévy process in $\mathbb{R}$:

$$L(t)a = tp(a) + W_a(t) + \int_{|\beta|\leqslant 1} \beta\, \tilde{N}_a(t, d\beta) + \int_{|\beta|>1} \beta\, N_a(t, d\beta),$$

where $p(a) \in \mathbb{R}$, W_a is Wiener in $\mathbb{R}$ and N is Poisson process.

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But no decomposition

- in cylindrical processes, as $a \mapsto \mathbb{1}_{[0,1]}(\|a\|)$ is not linear;
- in U, as L is not U-valued.

Examples of cylindrical Lèvy processes

Example: series approach

Theorem Let U be a Hilbert space with ONB $(e_k)_{k\in\mathbb{N}}$ and $(\sigma_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}$;

$(h_k)_{k\in\mathbb{N}}$ be a sequence of independent, real-valued Lévy processes.

1) (weak convergence) If for all $a \in U$ and $t \geqslant 0$ the sum

$$L(t)a := \sum_{k=1}^{\infty} \langle e_k, a\rangle \sigma_k h_k(t)$$

converges P-a.s. then it defines a cylindrical Lévy process $(L(t) : t \geqslant 0)$.

2) (strong convergence) If for all $t \geqslant 0$ the sum

$$L(t) := \sum_{k=1}^{\infty} e_k \, \sigma_k h_k(t)$$

converges P-a.s. then it defines an genuine Lévy process $(L(t) : t \geqslant 0)$.

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Example 0: for h_k standard, real-valued Brownian motion:

$(\sigma_k)_{k\in\mathbb{N}} \in \ell^\infty \iff$ cylindrical (Wiener) Lévy process

$(\sigma_k)_{k\in\mathbb{N}} \in \ell^2 \iff$ honest (Wiener) Lévy process

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Example 1: for h_k Poisson process with intensity 1:

$$(\sigma_k)_{k\in\mathbb{N}} \in \ell^2 \iff \text{cylindrical Lévy process}$$

$$(\sigma_k)_{k\in\mathbb{N}} \in \ell^1 \iff \text{honest Lévy process}$$

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converges P-a.s. then it defines a cylindrical Lévy process $(L(t) : t\geqslant 0)$.

Example 2: for h_k symmetric, standardised, α-stable:

$$(\sigma_k)_{k\in\mathbb{N}}\in\ell^{(2\alpha)/(2-\alpha)} \iff \text{cylindrical Lévy process}$$

$$(\sigma_k)_{k\in\mathbb{N}}\in\ell^{\alpha} \iff \text{honest Lévy process}$$

Example: subordination

Theorem

Let W be a cylindrical Wiener process in a Banach space U,
ℓ be a real-valued Lévy subordinator, independent of W.

Then, for each $t \geqslant 0$,

$$L(t) : U^* \to L^0_P(\Omega; \mathbb{R}), \qquad L(t)u^* = W(\ell(t))u^*$$

defines a cylindrical Lévy process $(L(t) : t \geqslant 0)$ in U.

Stochastic integration

Stochastic integration w.r.t cylindrical martingales

- M. Métivier, J. Pellaumail, 1980
- G. Kallianpur, J. Xiong, 1996
- R. Mikulevicius, B.L. Rozovskii, 1998.

Stochastic integration: classical case

Assume: Y classical Lévy process in a Hilbert space U

$$\Psi(s) := \mathbb{1}_{(a,b]}(s) \otimes \Phi \quad \text{for} \quad \Phi : \Omega \to \mathscr{L}_2(U,V).$$

Then $\displaystyle\int_0^T \Psi(s)\,dY(s) = \Phi\big(Y(b) - Y(a)\big)$

defines on $\mathscr{H}_0 = \{$ space of simple stochastic processes$\}$ the operator

$$I\colon \mathscr{H}_0 \to L_P^0(\Omega; V).$$

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Semimartingale decomposition: $Y(t) = M(t) + A(t)$ results in

$$I = I_M + I_A,$$

where $I_M\colon \mathscr{H}_0 \to L_P^2(\Omega; V)$

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where $I_M\colon \mathscr{H}_0 \to L_P^2(\Omega; V)$ extending by martingale properties

$I_A\colon \mathscr{H}_0 \to L_P^0(\Omega; V)$ extending by bounded variation

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$$\Psi(s) := \mathbb{1}_{(a,b]}(s) \otimes \Phi \quad \text{for} \quad \Phi : \Omega \to \mathscr{L}_2(U,V).$$

Then
$$\begin{aligned} \langle \int_0^T \Psi(s)\, dY(s), v\rangle &= \langle \Phi\big(Y(b) - Y(a)\big), v\rangle \\ &= \langle Y(b) - Y(a), \Phi^* v\rangle \end{aligned}$$

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&= \Big(L(b) - L(a)\Big)\big(\Phi^* v\big)
\end{aligned}$$

if $(L(t) : \, t \geqslant 0)$ is a cylindrical Lévy process in U.

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if $(L(t) : t \geqslant 0)$ is a cylindrical Lévy process in U.

Two problems:

- does there exists a random variable $J : \Omega \to V$ such that:

$$\langle J, v\rangle = \big(L(b) - L(a)\big)(\Phi^* v) \quad \text{for all } v \in V.$$

- Is the mapping $I\colon \mathscr{H}_0 \to L^0_P(\Omega; V)$ with $\Psi \mapsto \int_0^T \Psi(s)\, dL(s)$ continuous?

  no semimartingale decomposition of L in U

Radonifying the increments

Consider for fixed $0 \leqslant t_k \leqslant t_{k+1}$ a simple random variable

$$\Phi : \Omega \to \mathscr{L}_2(U, V) \qquad \Phi(\omega) := \sum_{i=1}^{n} \mathbb{1}_{A_i}(\omega)\varphi_i,$$

where $\varphi_i \in \mathscr{L}_2(U, V)$

$$A_i \in \mathscr{F}_{t_k} := \sigma\big(L(s)u : \; s \in [0, t_k], \, u \in U\big).$$

Since φ_i is Hilbert-Schmidt there exists $Z_i : \Omega \to V$ such that

$$\big(L(t_{k+1}) - L(t_k)\big)(\varphi_i^* v) = \langle Z_i, v\rangle \qquad \text{for all } v \in V.$$

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Define the V-valued random variable

$$\Phi\big(L(t_{k+1}) - L(t_k)\big) := \sum_{i=1}^{n} \mathbb{1}_{A_i} Z_i.$$

It satisfies for each $v \in V$:

$$\big\langle \Phi\big(L(t_{k+1}) - L(t_k)\big) \big\rangle\big\langle v \big\rangle = \sum_{i=1}^{n} \mathbb{1}_{A_i} \big(L(t_{k+1}) - L(t_k)\big)(\varphi_i^* v)$$

Radonifying the increments

Theorem: (with A. Jakubowski)
Let $0 \leqslant t_k \leqslant t_{k+1}$ be fixed. For each $\mathscr{F}_{t_k}$-measurable random variable

$$\Phi : \Omega \to \mathscr{L}_2(U, V),$$

there exists a random variable $Y : \Omega \to V$ and a sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ of simple random variables such that $\Phi_n \to \Phi$ P-a.s. and

$$Y = \lim_{n\to\infty} \Phi_n\big(L(t_{k+1}) - L(t_k)\big) \qquad \text{in probability.}$$

Define: $\Phi\big(L(t_{k+1}) - L(t_k)\big) := Y.$

Defining the stochastic integral

For a simple stochastic process of the form

$$\Psi : [0,T] \times \Omega \to \mathscr{L}_2(U,V), \qquad \Psi(t) = \sum_{j=0}^{N-1} \mathbb{1}_{(t_j,t_{j+1}]}(t)\Phi_j,$$

where $0 = t_0 < t_1 < \cdots < t_N = T$,

$\Phi_j : \Omega \to \mathscr{L}_2(U,V)$ is $\mathscr{F}_{t_j}$-measurable,

define the V-valued stochastic integral

$$I(\Psi) := \sum_{j=0}^{N-1} \Phi_j \big(L(t_{j+1}) - L(t_j) \big)$$

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Simple stochastic processes are dense in

$$\mathscr{H}(\mathscr{L}_2) := \big\{\Psi : \Omega \to D_-\big([0,T], \mathscr{L}_2(U,V)\big) : \text{ adapted}\big\},$$

where $D_-\big([0,T], \mathscr{L}_2(U,V)\big) := \{f : [0,T] \to \mathscr{L}_2(U,V) : \text{ càglàd}\}$,

equipped with the Skorokhod J_1-topology.

Defining the stochastic integral

Theorem: (with A. Jakubowski)
For every $\Psi \in \mathscr{H}(\mathscr{L}_2)$ there exists an V-valued random variable $I(\Psi)$ and a sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ of simple stochastic processes such that $\Psi_n \to \Psi$ P-a.s. in J_1 and

$$\int_0^T \Psi(s)\,dL(s) := \lim_{n\to\infty} I(\Psi_n) \qquad \text{in probability.}$$

Defining the stochastic integral

Theorem: (with A. Jakubowski)
For every $\Psi \in \mathscr{H}(\mathscr{L}_2)$ there exists an H-valued random variable $I(\Psi)$ and a sequence $\{\Psi_n\}_{n\in\mathbb{N}}$ of simple stochastic processes such that $\Psi_n \to \Psi$ P-a.s. in J_1 and

$$\int_0^T \Psi(s)\, dL(s) := \lim_{n\to\infty} I(\Psi_n) \qquad \text{in probability.}$$

Proof: Show that

(1) $\{I(\Psi_n) : n \in \mathbb{N}\}$ is tight

(2) for every $v \in V$ there exists a real-valued random variable Y_v such

$$\langle I(\Psi_n), v\rangle \to Y_v \text{ in probability}$$

Proof: $\{I(\Psi_n) : n \in \mathbb{N}\}$ is tight

Let $\Psi_n \to \Psi$ P-a.s. in J_1, where

$$\Psi_n(t) = \sum_{j=0}^{N_n - 1} \Phi_{n,j} \, \mathbb{1}_{(t_{n,j}, t_{n,j+1}]}(t).$$

$$\Phi_{n,j} : \left(\Omega, \mathscr{F}_{t_{n,j}}\right) \to \mathscr{L}_2(U,V)$$

$$0 = t_{n,0} \leqslant \cdots \leqslant t_{n,N_n} = T$$

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$0 = t_{n,0} \leqslant \cdots \leqslant t_{n,N_n} = T$

Then the stochastic integral $I(\Psi_n)$ is given by

$$I(\Psi_n) = J_{n,0} + \cdots + J_{n,N_n-1},$$

where $J_{n,j} := \Phi_{n,j}\big(L(t_{n,j+1}) - L(t_{n,j})\big) \colon \Omega \to V.$

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Define the regular conditional distribution $\mu_{n,j} : \mathfrak{B}(V) \times \Omega \to [0,1]$ by

$$\mu_{n,j}(B,\omega) : = P\Big(J_{n,j} \in B \,\Big|\, \mathscr{F}_{n,j}\Big)(\omega)$$

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$$= \Big(\underbrace{\lambda}_{\text{cylindrical Law of } L(1)} \circ \big(\Phi_{n,j}(\omega)\big)^{-1}\Big)^{t_{n,j+1}-t_{n,j}}(B)$$

Proof: $\{I(\Psi_n) : n \in \mathbb{N}\}$ is tight

Let $\big(J^*_{n,j} : j = 0, \ldots, N_n - 1,\ n \in \mathbb{N}\big)$ be r.v. on $(\widetilde{\Omega}, \widetilde{\mathscr{A}}, \widetilde{P})$ satisfying

- row-wise conditional independent under a σ-algebra $\mathscr{G} \subseteq \widetilde{A}$;
- $\widetilde{P}\big(J^*_{n,j} \in B \,\big|\, \mathscr{G}\big) = \widetilde{P}\big(J^*_{n,j} \in B \,\big|\, \mathscr{F}_{n,j}\big) = P\big(J_{n,j} \in B \,\big|\, \mathscr{F}_{n,j}\big),\ \ B \in \mathfrak{B}(V).$

Proof: $\{I(\Psi_n) : n \in \mathbb{N}\}$ **is tight**

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Define the (tangent) sum

$$I^*(\Psi_n) := J^*_{n,0} + \cdots + J^*_{n,N_n-1},$$

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$$I^*(\Psi_n) := J^*_{n,0} + \cdots + J^*_{n,N_n-1},$$

Theorem 2. *(Jakubowski 1988, Jakubowski, Riedle 2016)*

If $\displaystyle\sum_{k=0}^{N_{n-1}} J^*_{n,k}$ is tight then $\displaystyle\sum_{k=0}^{N_{n-1}} J_{n,k}$ is tight.

Proof: $\{I(\Psi_n) : n \in \mathbb{N}\}$ is tight

Let $\big(J^*_{n,j} : j = 0, \ldots, N_n - 1,\ n \in \mathbb{N}\big)$ be r.v. on $(\widetilde{\Omega}, \widetilde{\mathscr{A}}, \widetilde{P})$ satisfying

- row-wise conditional independent under a σ-algebra $\mathscr{G} \subseteq \widetilde{A}$;
- $\widetilde{P}\big(J^*_{n,j} \in B \,\big|\, \mathscr{G}\big) = \widetilde{P}\big(J^*_{n,j} \in B \,\big|\, \mathscr{F}_{n,j}\big) = P\big(J_{n,j} \in B \,\big|\, \mathscr{F}_{n,j}\big), \quad B \in \mathfrak{B}(V).$

Define the (tangent) sum

$$I^*(\Psi_n) := J^*_{n,0} + \cdots + J^*_{n,N_n-1},$$

and its regular conditional distribution $\sigma^*_n : \mathfrak{B}(V) \times \Omega \to [0,1]$ by

$$\sigma^*_n(B, \omega) : = \widetilde{P}\Big(I^*(\Psi_n) \in B \;\Big|\; \mathscr{G}\Big)(\omega)$$

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&= \Big(\lambda \circ \big(\Phi_{n,0}(\omega)\big)^{-1}\Big)^{t_{n,1}-t_{n,0}} * \cdots * \Big(\lambda \circ \big(\Phi_{n,N_n-1}(\omega)\big)^{-1}\Big)^{t_{n,N_n}-t_{n,N_n-1}}(B)
\end{aligned}$$

Proof: $\{I(\Psi_n) : n \in \mathbb{N}\}$ **is tight**

For $\varepsilon > 0$ there exists a compact set $K \subseteq \mathscr{L}_2(U, V)$ such that

$$A_K := \{\omega \in \Omega : \Phi_{n,j}(\omega) \in K \text{ for all } j = 0, \ldots, N_n - 1, n \in \mathbb{N}\}$$

satisfies $P(A_K^c) \leqslant \varepsilon$.

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$$\Big\{\sigma_n^*(\cdot, \omega) : n \in \mathbb{N}\Big\}$$
$$\subseteq \Big\{(\lambda \circ \varphi_1^{-1})^{*s_1} * \cdots * (\lambda \circ \varphi_n^{-1})^{*s_n} : \varphi_j \in K,\, s_1 + \cdots + s_n = T,\, n \in \mathbb{N}\Big\}$$

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Since $P_{I^*(\Psi_n)} = E[\sigma_n^*]$ we have

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which implies tightness of $\{I(\Psi_n) : n \in \mathbb{N}\}$.

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