On weak approximation of BSDEs driven by Lévy processes

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Joint work with **D. Madan** and **M. Stadje**

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Introduction

- Backward stochastic differential equations (BSDEs) have turned up in a range of different setting, notably in many applications in mathematical finance and as non-linear expectations and risk-measures.
- Unlike in the case of BSDEs without jumps, exact sampling methods from the probability distribution of the increments of the driving Poisson random measures are in general not readily available, which is an issue in the practical implementation of approximation schemes.
- Motivated by this observation, we propose a weak approximation scheme for BSDEs driven by a Wiener process and independent Poisson random measure (allowing the approximating processes to be defined on filtrations that are different from the one the BSDE lives on).
- We provide a functional weak limit theorem for the discrete-time BSDEs.

BSDEs: setting

- W: d_1-dimensional Wiener process and an independent
- X: d_2-dimensional Lévy process of the form

$$X_t = \int_{[0,t]\times\mathbb{R}^{d_2}\setminus\{0\}} x(N(\mathrm{d}s\times\mathrm{d}x) - \nu(\mathrm{d}x)\mathrm{d}s) = \int_{[0,t]\times\mathbb{R}^{d_2}\setminus\{0\}} x\tilde{N}(\mathrm{d}s\times\mathrm{d}x),$$

Consider BSDEs of the form:

$$Y_t = F + \int_t^T g(s, Y_s, Z_s, \tilde{Z}_s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \quad (1)$$
$$- \int_{(t,T]\times\mathbb{R}^{d_2}\setminus\{0\}} \tilde{Z}_s(x)\tilde{N}(\mathrm{d}s\times\mathrm{d}x), \quad t \in [0, T],$$

with *driver function*

$$g : [0, T] \times \mathbb{R} \times \mathbb{R}^{d_1} \times L^2(\nu(\mathrm{d}x), \mathcal{B}(\mathbb{R}^{d_2}\setminus\{0\})) \to \mathbb{R}.$$

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BSDE: setting

Assumption

(i) g is continuous as function of $t \in [0, T]$ at any $(y, z, \tilde{z})$, and

(ii) g is Lipschitz continuous in $(y, z, \tilde{z})$ uniformly for all $t \in [0, T]$, that is, there exists a positive K satisfying

$$|g(t, y_1, z_1, \tilde{z}_1) - g(t, y_0, z_0, \tilde{z}_0)| \leq K\Bigg(|y_1 - y_0| + |z_1 - z_0| + \sqrt{\int_{\mathbb{R}^{d_2}\setminus\{0\}} |\tilde{z}_1(x) - \tilde{z}_0(x)|^2 \nu(\mathrm{d}x)}\Bigg), \quad (2)$$

Under this Assumption, the BSDE (1) has a unique solution (Tang & Li (1994) and Royer (2006)).

Question:

1. Random walk approximation schemes for BSDE (3)?
2. Weak convergence (functional)?

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BSΔEs driven by random walks

- $\pi_N := \{t_0, t_1, \ldots, t_N\}$ with $t_i = iT/N$, $\Delta = T/N$.
- We assume that $W^{(\pi)}$ and $X^{(\pi)}$ are independent, square-intergrable martingales defined on the probability space $(\Omega, \mathcal{F}^{(\pi)}, \mathbb{P})$ which are piecewise constant on $[t_i, t_{i+1})$

Let $W^{(\pi)}$ vector of zero-mean random walks with $\Delta W_{t_i}^{(\pi)} := W_{t_{i+1}}^{(\pi)} - W_{t_i}^{(\pi)}$ satisfying

$$\mathbb{E}_{t_i}\left[\left(\Delta W_{t_i}^{(\pi)}\right)\left(\Delta W_{t_i}^{(\pi)}\right)'\right] = \Delta I_{d_1}, \qquad i = 0, \ldots, N-1,$$

$$\sup_\pi \mathbb{E}[|W_T^{(\pi)}|^{2+\epsilon}] < \infty, \qquad \text{for some } \epsilon > 0,$$

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Convergence of BSΔEs to BSDEs: Setting

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$$\Delta^{-1/2}\mathbb{E}[|\Delta X^{(\pi)}_{t_i}|] \longrightarrow 0, \qquad \Delta \to 0, \qquad \text{and}$$

$$\Delta^{-1}\mathbb{E}_{t_i}\left[\left(\Delta X^{(\pi)}_{t_i}\right)\left(\Delta X^{(\pi)}_{t_i}\right)'\right] \longrightarrow (\nu_{k,l})^{d_2}_{k,l=1},$$

$$\nu_{k,l} = \int h_k(x)h_l(x)\nu(\mathrm{d}x), \qquad h_k(x) = x_k, \quad k = 1,\ldots,d_2, \ \text{and}$$

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It is also assumed that the step-size distribution $G^{(\pi)}$ satisfies

$$\int_{\mathbb{R}^{d_2}\setminus\{0\}} g(x)\nu^{(\pi)}(\mathrm{d}x) \longrightarrow \int_{\mathbb{R}^{d_2}\setminus\{0\}} g(x)\nu(\mathrm{d}x),$$

$$\text{as } \Delta \to 0, \text{ with} \qquad \nu^{(\pi)}(\mathrm{d}x) := \Delta^{-1}G^{(\pi)}(\mathrm{d}x),$$

for continuous bounded $g : \mathbb{R}^d\setminus\{0\} \to \mathbb{R}$ that are 0 around $x = 0$ and have a limit as $|x| \to \infty$. Finally, assume

$$\liminf_{\Delta\to 0} \mathbb{P}\left(\Delta X^{(\pi)}_{t_i} = 0\right) \geq a, \qquad \text{for some } a > 0. \tag{3}$$

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Convergence of BSΔEs to BSDEs: Setting

- As $\Delta \to 0$ we have

$$(W^{(\pi)}, X^{(\pi)}) \xrightarrow{\mathcal{L}} (W, X), \tag{4}$$

  where $\xrightarrow{\mathcal{L}}$ denotes weak-convergence in the Skorokhod J_1-topology.
- The condition (3) is needed for stability (as we will see later).

BSDEs driven by random walks

Consider the following BSΔE:

$$\begin{aligned} Y_{t_i}^{(\pi)} &= F^{(\pi)} + \sum_{j=i}^{N-1} g^{(\pi)}(t_j, Y_{t_j}^{(\pi)}, Z_{t_j}^{(\pi)}, \tilde{Z}_{t_j}^{(\pi)})\Delta - \sum_{j=i}^{N-1} Z_{t_j}^{(\pi)} \Delta W_{t_j}^{(\pi)} \\ &- \sum_{j=i}^{N-1} \left\{ \tilde{Z}_{t_j}^{(\pi)}(\Delta X_{t_j}^{(\pi)}) I_{\{\Delta X_{t_j}^{(\pi)} \neq 0\}} - \mathbb{E}_{t_j}\left[\tilde{Z}_{t_j}^{(\pi)}(\Delta X_{t_j}^{(\pi)}) I_{\{\Delta X_{t_j}^{(\pi)} \neq 0\}} \right] \right\} \\ &- \left(M_T^{(\pi)} - M_{t_i}^{(\pi)} \right), \end{aligned}$$

The BSΔE can be equivalently expressed in differential notation as

$$\begin{aligned} \Delta Y_{t_i}^{(\pi)} &= -g^{(\pi)}(t_i, Y_{t_i}^{(\pi)}, Z_{t_i}^{(\pi)}, \tilde{Z}_{t_i}^{(\pi)})\Delta + Z_{t_i}^{(\pi)} \Delta W_{t_i}^{(\pi)} \\ &+ \left\{ \tilde{Z}_{t_i}^{(\pi)}(\Delta X_{t_i}^{(\pi)}) I_{\{\Delta X_{t_i}^{(\pi)} \neq 0\}} - \mathbb{E}_{t_i}\left[\tilde{Z}_{t_i}^{(\pi)}(\Delta X_{t_i}^{(\pi)}) I_{\{\Delta X_{t_i}^{(\pi)} \neq 0\}} \right] \right\} \\ &+ \Delta M_{t_i}^{(\pi)}, \\ Y_T^{(\pi)} &= F^{(\pi)}, \end{aligned}$$

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BSΔEs

Proposition

For $\Delta < 1/K$ the BSΔE has a unique solution $(Y^{(\pi)}, Z^{(\pi)}, \tilde{Z}^{(\pi)}, M^{(\pi)})$, which satisfies the relations: for $t_i \in \pi$,

$$
\begin{aligned}
Y^{(\pi)}_{t_i} &= g^{(\pi)}(t_i, Y^{(\pi)}_{t_i}, Z^{(\pi)}_{t_i}, \tilde{Z}^{(\pi)}_{t_i})\Delta + \mathbb{E}_{t_i}[Y^{(\pi)}_{t_{i+1}}] \\
&= \mathbb{E}_{t_i}\left[F^{(\pi)} + \sum_{j=i}^{N-1} g^{(\pi)}(t_j, Y^{(\pi)}_{t_j}, Z^{(\pi)}_{t_j}, \tilde{Z}^{(\pi)}_{t_j})\Delta\right], \\
Z^{(\pi)}_{t_i} &= \Delta^{-1}\,\mathbb{E}_{t_i}\left[Y^{(\pi)}_{t_{i+1}}\Delta W^{(\pi)}_{t_i}\right], \\
\tilde{Z}^{(\pi)}_{t_i}(x) &= \mathbb{E}_{t_i}\left[Y^{(\pi)}_{t_{i+1}}|\Delta X^{(\pi)}_{t_i} = x\right] - \mathbb{E}_{t_i}\left[Y^{(\pi)}_{t_{i+1}}|\Delta X^{(\pi)}_{t_i} = 0\right], \\
\Delta M^{(\pi)}_{t_i} &= Y^{(\pi)}_{t_{i+1}} - \mathbb{E}_{t_i}\left[Y^{(\pi)}_{t_{i+1}}\right] - Z^{(\pi)}_{t_i}\Delta W^{(\pi)}_{t_i} \\
&\quad - \left\{\tilde{Z}^{(\pi)}_{t_i}(\Delta X^{(\pi)}_{t_i}) I_{\{\Delta X^{(\pi)}_{t_i} \neq 0\}} - \mathbb{E}_{t_i}\left[\tilde{Z}^{(\pi)}_{t_i}(\Delta X^{(\pi)}_{t_i}) I_{\{\Delta X^{(\pi)}_{t_i} \neq 0\}}\right]\right\}.
\end{aligned}
$$

BSΔEs

Proposition

If $F^{(\pi)}$ *is independent of* $W^{(\pi)}$ *then* $M^{(\pi)} \equiv 0$.

In particular, it follows that in the pure jump case, the martingale $M^{(\pi)}$ is zero and the representation property holds true.

Stability of the BSΔEs

We impose the following conditions on the approximating driver functions $g^{(\pi)}$

Assumption

(i) For some $K > 0$, the drivers $g^{(\pi)}$ are uniformly K-Lipschitz continuous.
(ii) $g^{(\pi)}(t, 0, 0, 0)$ is bounded uniformly over all $t \in \pi$ and partitions π.
(iii) For every $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^{d_1}$ and uniformly Lipschitz continuous function $\tilde{z}$ (*i.e.*, $\tilde{z}$ for which $|\tilde{z}(x)|/|x|$ is bounded over all $x \in \mathbb{R}^{d_2} \setminus \{0\}$), we have

$$\lim_{\Delta \to 0} g^{(\pi)}(t, y, z, \tilde{z}) = g(t, y, z, \tilde{z}). \tag{5}$$

Stability of the BSΔEs

Theorem

There exists an $n_0 \in \mathbb{N}$ and a constant $\bar{C}$ such that for all $\pi = \pi_N$ with $N \geq n_0$, all drivers $f^{(\pi),0}, f^{(\pi),1}$ satisfying Assumption 1(i)-(ii), and square integrable terminal conditions $F^{(\pi),0}, F^{(\pi),1}$, and $t_i \in \pi$, we have

$$\mathbb{E}\left[\max_{t_j \leq t_i, t_j \in \pi} |\delta Y_{t_j}^{(\pi)}|^2 + \sum_{j=0}^{i-1} \left\{ |\delta Z_{t_j}^{(\pi)}|^2 \Delta + |\delta M_{t_j}^{(\pi)}|^2 \right.\right.$$
$$\left.\left. + |\delta \tilde{Z}_{t_j}^{(\pi)}(\Delta X_{t_j}^{(\pi)}) - \mathbb{E}_{t_j}[\delta \tilde{Z}_{t_j}^{(\pi)}(\Delta X_{t_j}^{(\pi)})]|^2 \right\}\right]$$
$$\leq \bar{C}\mathbb{E}\left[|\delta Y_{t_i}^{(\pi)}|^2 + \sum_{j=0}^{i-1} |\delta f^{(\pi)}(t_j, Y_{t_j}^{(\pi),0}, Z_{t_j}^{(\pi),0}, \tilde{Z}_{t_j}^{(\pi),0})|^2 \Delta \right],$$

with $\delta Y^{(\pi)} = Y^{(\pi),0} - Y^{(\pi),1}$, etc.

Remark

The condition (3) that $X_{t_i}^{(\pi)}$ has a uniformly positive probability of being zero plays an important role in the proof.

Stability of the BSDEs

Remark

In continuous time the following analogous estimate holds true for some constant $\bar{c} > 0$:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le t'} |\delta Y_t|^2 + \int_0^{t'} |\delta Z_s|^2 \mathrm{d}s + \int_{[0,t']\times\mathbb{R}^{d_2}\setminus\{0\}} |\delta \tilde{Z}_s(x)|^2 \nu(\mathrm{d}x)\mathrm{d}s\right] (6)\\
&\le \quad \bar{c}\,\mathbb{E}\left[|\delta Y_{t'}|^2 + \int_0^{t'} |\delta f(s, Y_s^0, Z_s^0, \tilde{Z}_s^0)|^2 \mathrm{d}s\right], \qquad t' \in [0, T],
\end{aligned}$$

where $\delta Y = Y^1 - Y^0$ etc. For a proof of (6), see for instance to Proposition 3.3 in Becherer (2006) or Lemma 3.1.1 in Delong (2013).

Convergence of BSΔEs to BSDEs

Theorem

Let the Assumption hold and let (π) be a sequence of partitions π with the mesh Δ tending to zero. If $F^{(\pi)}$ converges to F in L^2, then $Y^{(\pi)} \xrightarrow{\mathcal{L}} Y$ and in particular

$$Y_0^{(\pi)} \to Y_0.$$

Moreover, with d_S denoting the Skorokhod metric, we have

$$\mathbb{E}[d_S^2(Y^{(\pi)}, Y)] \to 0.$$

Elements of the proof 1

The idea of the proof, inspired by Briand *et al.* (2001,2002), is to reduce the question of weak convergence of the solutions of the BSΔEs to the solution of BSDE to that of the Picard sequences by using the fact that both the solutions of the BSDE and of the BSΔEs are equal to limits of Picard sequences.

Elements of the proof 2

In the proof we deploy the notions of extended weak convergence and weak convergence of filtrations (see Coquet *et al.* (2004) and Mémin (2003)):

Definition

Given stochastic processes $Z = (Z_t)_{t\in[0,T]}$ and $(Z^n)_{n\in\mathbb{N}}$ with $Z^n = (Z^n_t)_{t\in[0,T]}$ defined on filtered probability spaces $(\Omega, \Gamma, (\Gamma_t), \mathbb{P})$ and $(\Omega, \Gamma^n, (\Gamma^n_t), \mathbb{P})$ respectively, we say (i) Γ^n weakly converges to Γ [denoted $\Gamma^n \xrightarrow{w} \Gamma$] if for every $A \in \Gamma$ the sequence of processes $(\mathbb{E}[I_A|\mathcal{G}^n_t])_{t\in[0,T]}$ converges to the process $(\mathbb{E}[I_A|\mathcal{F}_t])_{t\in[0,T]}$ and (ii) (Z^n, Γ^n) weakly converges to (Z, Γ) [denoted $(Z^n, \Gamma^n) \xrightarrow{w} (Z, \Gamma)$] if for every $A \in \Gamma$ the sequence of processes $(Z^n_t, \mathbb{E}[I_A|\mathcal{G}^n_t])_{t\in[0,T]}$ converges to the process $(Z_t, \mathbb{E}[I_A|\mathcal{G}_t])_{t\in[0,T]}$. In both cases the convergence is in probability under the Skorokhod J_1-topology (on the space D of càdlàg functions).

We have:

Proposition (Proposition 2, Mémin (2003))

We have $((W^{(\pi)}, X^{(\pi)}), \mathcal{F}^{(\pi)}) \xrightarrow{w} ((W, X), \mathcal{F})$ *as* $\Delta \to 0$*. In particular,* $\mathcal{F}^{(\pi)} \xrightarrow{w} \mathcal{F}$*.*

Application: convergence of spectral risk measures

Definition

A *dynamic coherent risk measure* $\rho = (\rho_t)_{t\in I}$ is a map $\rho : \mathcal{L}^2 \to \mathcal{S}^2(\mathcal{I})$ that satisfies the following properties:

(i) *(cash invariance)* for $m \in \mathcal{L}_t^2$, $\rho_t(X+m) = \rho_t(X) - m$;

(ii) *(monotonicity)* for $X, Y \in \mathcal{L}^2$ with $X \geq Y$, $\rho_t(X) \leq \rho_t(Y)$;

(iii) *(positive homogeneity)* for $X \in \mathcal{L}^2$ and $\lambda \in \mathcal{L}_t^\infty$, $\rho_t(|\lambda|X) = |\lambda|\rho_t(X)$;

(iv) *(subadditivity)* for $X, Y \in \mathcal{L}^2$, $\rho_t(X+Y) \leq \rho_t(X) + \rho_t(Y)$.

Definition

A dynamic coherent risk measure ρ is *(strongly) time-consistent* if either of the following holds:

(v) *(strong time-consistency)* for $X, Y \in \mathcal{L}^2$ and s, t with $s \leq t$, $\rho_t(X) \leq \rho_t(Y) \Rightarrow \rho_s(X) \leq \rho_s(Y)$;

(vi) *(recursiveness)* for $X \in \mathcal{L}^2$ and s, t with $s \leq t$, $\rho_s(\rho_t(X)) = \rho_s(X)$.

Note: (vi) can be used to define a risk measure recursively on a finite set

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Spectral risk measures

Let $\mathrm{C}^{\Psi\circ\mathbb{P}}$ be the Choquet integral given by

$$\mathrm{C}^{\Psi\circ\mathbb{P}}(X) = \int_0^{\infty} (\Psi \circ \mathbb{P})(X > x)\mathrm{d}x - \int_{-\infty}^{0} (1 - (\Psi \circ \mathbb{P})(X > x))\mathrm{d}x, \quad (7)$$

where $\Psi : [0, 1] \to [0, 1]$ is a concave increasing continuous function (satisfying some integrability conditions near 0 and 1).

Definition

(i) The conditional Choquet-type integral $\mathrm{C}^{\Psi\circ\mathbb{P}}(\,\cdot\,|\Phi_t) : \mathcal{L}^2 \to \mathcal{L}^2_t$ is

$$\mathrm{C}^{\Psi\circ\mathbb{P}}(X|\mathcal{F}_t) := \int_{\mathbb{R}_+} \Psi\left(\mathbb{P}(X^+ > x|\mathcal{F}_t)\right)\mathrm{d}x - \int_{\mathbb{R}_+} \Psi\left(\mathbb{P}(X^- > x|\mathcal{F}_t)\right)\mathrm{d}x$$

(ii) $\mathrm{C}^{\Psi\circ\mathbb{P}}(-X|\mathcal{F}_t)$ is the spectral risk measure corresponding to Ψ.

Note: on a given time-grid a (time-consistent) dynamic spectral risk-measures may be defined recursively.

Question: limit as the mesh of the grid tends to zero?

Spectral risk measures

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Dynamic risk measures

Let $\mathcal{I} = [0, T]$ or equal to a finite grid. Dynamic coherent risk-measures $\rho(X)$ are defined as solutions to the BSDEs/BSΔEs with final condition $-X$ and driver functions satisfying:

Definition

For a given Borel measure μ on $\mathbb{R}^k \backslash \{0\}$ we call a function $g : \mathcal{I} \times \mathcal{L}^2(\mu) \to \mathbb{R}$ a *driver function* if, for any $z \in \mathcal{L}^2(\mu)$, $t \mapsto g(t, z)$ is continuous (in case $\mathcal{I} = [0, T]$) and the following holds:

(i) *(Lipschitz-continuity)* for any $t \in \mathcal{I}$, $z_1, z_2 \in \mathcal{L}^2(\mu)$

$$|g(t, z_1) - g(t, z_2)| \leq K|z_1 - z_2|_{2,\mu}.$$

A driver function g is called *coherent* if the following hold:

(ii) *(positive homogeneity)* for any $r \in \mathbb{R}_+$, $t \in \mathcal{I}$, $z \in \mathcal{L}^2(\mu)$

$$g(t, rz) = rg(t, z);$$

(iii) *(subadditivity)* for $t \in \mathcal{I}$, $z_1, z_2 \in \mathcal{L}^2(\mu)$

$$g(t, z_1 + z_2) \leq g(t, z_1) + g(t, z_2).$$

Dynamic spectral risk measures (continuous time)

Restricting to the pure-jummp setting, we define dynamic spectral risk-measures in continuous-time as dynamic coherent risk-measures associated to the following type of driver functions:

Definition

The *spectral driver function* $\bar{g} : \mathcal{L}^2(\nu) \to \mathbb{R}_+$ is

$$\bar{g}(u) := \mathsf{C}^{\Gamma_+ \circ \nu}(u^+) + \mathsf{C}^{\Gamma_- \circ \nu}(u^-)$$

where ν is the Lévy measure and

$$\mathsf{C}^{\Gamma \circ \nu}(u) = \int_0^\infty \nu(\Gamma(u > x))\mathrm{d}x,$$

where Γ_+, Γ_- are increasing, concave funnctions satisfying

$$\int_{(0,\nu(\mathbb{R}))} [\frac{\Gamma(y)}{y\sqrt{y}}]\mathrm{d}y < \infty.$$

Convergence of spectral risk measures

Ingredients:

- π: a uniform grid with mesh Δ
- F : path-functional that is continuous in the Skorokhod J_1-topology and and such that for some $k \in \mathbb{R}_+$

$$|F(\omega)| \leq k\|\omega\|_\infty \text{ for all } \omega \in \mathbb{D}([0,T],\mathbb{R}^k),$$

- S^Δ: an iterated spectral risk-measure on π corresponding distortion Ψ_Δ.

In order to obtain a non-degenerate limit for $S^\Delta(F(X^{(\pi)})$ as $\Delta \searrow 0$, Ψ_Δ need to be scaled as follows, uniformly in p:

$$\Psi_\Delta(p) = p + \Delta\left\{\Gamma_+(p/\Delta)I_{[0,\frac{1}{2}]}(p) + \Gamma_-((1-p)/\Delta)I_{(\frac{1}{2},1]}(p)\right\} + o(\Delta),$$

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Convergence of spectral risk measures

Theorem

$$\mathsf{S}^{\Delta}\left(F\left(\tilde{X}^{(\pi)}\right)\right) \xrightarrow{\mathcal{L}} \rho^{\bar{g}}\left(F\left(X\right)\right), \quad \Delta \searrow 0.$$

Remark

Given two concave probability distortions Ψ_+ and Ψ_- satisfying the integrability condition (**??**) (with $\mu(\mathbb{U}) = 1$) one may explicitly construct a sequence $(\Psi_\Delta)_{\Delta\in(0,1]}$ satisfying Definition **??** as follows:

$$\Psi_\Delta(p) = p + (\Gamma_+(p/\Delta) I_{[0,\frac{1}{2}]}(p) + \Gamma_-((1-p)/\Delta) I_{(\frac{1}{2},1]}(p))\,\Delta, \qquad p \in [0,1],$$

where, inspired by Eberlein *et al.* (2014), we suppose that the functions $\Gamma_+, \Gamma_- : \mathbb{R}_+ \to \mathbb{R}_+$ are given by

$$\Gamma_+(x) = a\,\Psi_+(1 - \mathrm{e}^{-cx}), \qquad \Gamma_-(x) = \frac{b}{d}\,\Psi_-(1 - \mathrm{e}^{-dx}), \qquad x \in \mathbb{R}_+,$$

for some a, b, c and $d \in \mathbb{R}_+ \setminus \{0\}$ satisfying the restrictions

$$\Gamma_+(1/(2\Delta)) = \Gamma_-(1/(2\Delta)) < 1/(2\Delta), \quad b\,\Psi'_-(0^+) \in (0,1), \qquad (8)$$

where $f'(0^+)$ denote the right-derivative of a function f at $x = 0$.
It is straightforward to check that, for any $\Delta \in (0,1]$, Ψ_Δ is a concave probability distortion (the first condition in (8) guarantees continuity at $p = 1/2$ and $\Psi_\Delta(1/2) < 1$) and that $\Gamma_-(x) \le x$ for any $x \in \mathbb{R}_+$

Remark

Examples of funtionals F that are continuous in the Skorokhod topology and satisfy condition (**??**) include (a) a European call option payoff with strike $K \in \mathbb{R}_+$ ($F(\omega) = (\omega(T) - K)^+$); (b) the time-average ($F(\omega) = \frac{1}{T}\int_0^T \omega(s)\mathrm{d}s$) and (c) the running maximum ($F(\omega) = \sup_{s\in[0,T]} \omega(s)$).

Remark

We note that Υ_Δ may be equivalently expressed in terms of Ψ_Δ and $\widehat{\Psi}_\Delta$ as follows:

$$\Upsilon_\Delta = \sup_{x\in(0,\frac{1}{2}]} \left| \frac{\Psi_\Delta(x) - x}{\Gamma_+(x/\Delta)\Delta} - 1 \right| \bigvee \sup_{x\in(0,\frac{1}{2})} \left| \frac{x - \widehat{\Psi}_\Delta(x)}{\Gamma_-(x/\Delta)\Delta} - 1 \right| .$$