Bernstein-gamma functions and exponential functionals of Lévy processes

M. Savov[1]
joint work with P. Patie[2]

ICLP8, Angers, France

[1]Marie Sklodowska Curie Individual Fellowship at IMI, BAS, 2015-2017-project MOCT
[2]Cornell University

ϕ is a Bernstein function that is $\phi \in \mathcal{B}$ iff

$$\phi(\mathrm{z}) = \mathrm{m} + \delta \mathrm{z} + \int_0^\infty \left(1 - \mathrm{e}^{-\mathrm{zy}}\right) \mu(\mathrm{dy}),$$

where $\mathrm{m}, \delta \geq 0$; $\int_0^\infty (1 \wedge \mathrm{y})\, \mu(\mathrm{dy}) < \infty$.

ϕ is a Bernstein function that is $\phi \in \mathcal{B}$ iff

$$\phi(\mathrm{z}) = \mathrm{m} + \delta\mathrm{z} + \int_0^\infty \left(1 - \mathrm{e}^{-\mathrm{zy}}\right)\mu(\mathrm{dy}),$$

where $\mathrm{m}, \delta \geq 0$; $\int_0^\infty (1 \wedge \mathrm{y})\,\mu(\mathrm{dy}) < \infty$.

The unique solution to

$$W_\phi(\mathrm{z}+1) = \phi(\mathrm{z}) W_\phi(\mathrm{z}) \text{ on } \mathrm{z} \in \mathbb{C}_{(0,\infty)} = \{\mathrm{z} \in \mathbb{C} : \mathrm{Re}(\mathrm{z}) > 0\} \tag{0.1}$$

in the space of Mellin transforms of positive random variables we call a Bernstein-gamma function.

ϕ is a Bernstein function that is $\phi \in \mathcal{B}$ iff

$$\phi(\mathrm{z}) = \mathrm{m} + \delta \mathrm{z} + \int_0^\infty \left(1 - \mathrm{e}^{-\mathrm{zy}}\right) \mu(\mathrm{dy}),$$

where $\mathrm{m}, \delta \geq 0$; $\int_0^\infty (1 \wedge \mathrm{y}) \, \mu(\mathrm{dy}) < \infty$.

The unique solution to

$$\mathrm{W}_\phi\left(\mathrm{z}+1\right) = \phi(\mathrm{z}) \mathrm{W}_\phi\left(\mathrm{z}\right) \text{ on } \mathrm{z} \in \mathbb{C}_{(0,\infty)} = \{\mathrm{z} \in \mathbb{C} : \ \mathrm{Re}(\mathrm{z}) > 0\} \qquad (0.1)$$

in the space of Mellin transforms of positive random variables we call a Bernstein-gamma function.

Recall that the Mellin transform of a positive random variable Y is formally given by $\mathcal{M}_\mathrm{Y}(\mathrm{z}) = \mathbb{E}\left[\mathrm{Y}^{\mathrm{z}-1}\right]$.

1. Understanding of W_ϕ as a meromorphic/holomorphic function
2. Development of Stirling type of asymptotic

1. W_ϕ appears crucially in the spectral studies of the generalized Laguerre semigroups and the positive self-similar Markov processes as instances of non-selfadjoint Markov semigroups. The quantification of its analytic properties offers explicit information about eigen- and coeigen-functions, their norms, etc.
2. W_ϕ are related to the "phenomenon of self-similarity" the same way the Gamma function appears in some diffusions
3. Amongst W_ϕ are some well-known special functions, e.g. the Barnes-Gamma function, the q-gamma function
4. W_ϕ appears in exponential functionals of Lévy processes

1. W_ϕ appears crucially in the spectral studies of the generalized Laguerre semigroups and the positive self-similar Markov processes as instances of non-selfadjoint Markov semigroups. The quantification of its analytic properties offers explicit information about eigen- and coeigen-functions, their norms, etc.
2. W_ϕ are related to the “phenomenon of self-similarity” the same way the Gamma function appears in some diffusions
3. Amongst W_ϕ are some well-known special functions, e.g. the Barnes-Gamma function, the q-gamma function
4. W_ϕ appears in exponential functionals of Lévy processes

1. W_ϕ appears crucially in the spectral studies of the generalized Laguerre semigroups and the positive self-similar Markov processes as instances of non-selfadjoint Markov semigroups. The quantification of its analytic properties offers explicit information about eigen- and coeigen-functions, their norms, etc.
2. W_ϕ are related to the "phenomenon of self-similarity" the same way the Gamma function appears in some diffusions
3. Amongst W_ϕ are some well-known special functions, e.g. the Barnes-Gamma function, the q-gamma function
4. W_ϕ appears in exponential functionals of Lévy processes

1. W_ϕ appears crucially in the spectral studies of the generalized Laguerre semigroups and the positive self-similar Markov processes as instances of non-selfadjoint Markov semigroups. The quantification of its analytic properties offers explicit information about eigen- and coeigen-functions, their norms, etc.
2. W_ϕ are related to the "phenomenon of self-similarity" the same way the Gamma function appears in some diffusions
3. Amongst W_ϕ are some well-known special functions, e.g. the Barnes-Gamma function, the q-gamma function
4. W_ϕ appears in exponential functionals of Lévy processes

Denote by

$$\overline{\mathcal{N}} = \left\{ \Psi : \Psi(z) = \frac{\sigma^2}{2} z^2 + bz + \int_{-\infty}^{\infty} \left(e^{zr} - 1 - zr1_{|r|<1} \right) \Pi(dr) - q \right\}$$

the set of all Lévy-Khintchine exponents of possibly killed at exponential random time of parameter $q \geq 0$ Lévy processes.

Denote by

$$\overline{\mathcal{N}} = \left\{ \Psi :\ \Psi(z) = \frac{\sigma^2}{2} z^2 + bz + \int_{-\infty}^{\infty} \left(e^{zr} - 1 - zr1_{|r|<1} \right) \Pi(dr) - q \right\}$$

the set of all Lévy-Khintchine exponents of possibly killed at exponential random time of parameter $q \geq 0$ Lévy processes.

The random variables

$$I_\Psi = \int_0^{e_q} e^{-\xi_s} ds,\ e_q \sim \mathrm{Exp}(q); e_0 = \infty$$

are called exponential functionals of Lévy processes and

$$I_\Psi < \infty \iff \Psi \in \mathcal{N} = \left\{ \Psi \in \overline{\mathcal{N}} :\ q > 0 \text{ or } \lim_{s \to \infty} \xi_s = \infty \right\} \subsetneq \overline{\mathcal{N}}.$$

1. For any $\Psi \in \overline{\mathcal{N}}$ to solve and characterize the solutions of

$$f(z+1) = \frac{-z}{\Psi(-z)} f(z) \text{ on } \{z \in i\mathbb{R} : \Psi(-z) \neq 0\} \tag{0.2}$$

in terms of the global quantities of Ψ

2. Use that $\mathcal{M}_{I_\Psi}(z+1) = \mathbb{E}\left[I_\Psi^z\right]$ solves in some sense (0.2) to obtain information about I_Ψ

1. For any $\Psi \in \overline{\mathcal{N}}$ to solve and characterize the solutions of

$$\mathrm{f}\left(\mathrm{z}+1\right) = \frac{-\mathrm{z}}{\Psi(-\mathrm{z})}\mathrm{f}(\mathrm{z}) \text{ on } \left\{\mathrm{z} \in i\mathbb{R} : \Psi(-\mathrm{z}) \neq 0\right\} \tag{0.2}$$

   in terms of the global quantities of Ψ

2. Use that $\mathcal{M}_{\mathrm{I}_\Psi}\left(\mathrm{z}+1\right) = \mathbb{E}\left[\mathrm{I}_\Psi^{\mathrm{z}}\right]$ solves in some sense (0.2) to obtain information about I_Ψ

1. Appear in financial and insurance mathematics; branching with immigration; fragmentation; self-similar growth fragmentation; etc.
2. We also use it in our work on the spectral theory of positive self-similar semigroups

1. Appear in financial and insurance mathematics; branching with immigration; fragmentation; self-similar growth fragmentation; etc.
2. We also use it in our work on the spectral theory of positive self-similar semigroups

1. I_ψ introduced and studied by Urbanik when ξ is a subordinator
2. Further studied by Carmona, Petit and Yor who have in special cases $\mathcal{M}_{I_\Psi}(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_{I_\Psi}(z)$
3. Maulik and Zwart derive this key recurrent equation in some generality and utilize it
4. Kuznetsov solves this recurrent relation for some classes of Lévy processes
5. There are various other contributions relying on different approaches

1. I_Ψ introduced and studied by Urbanik when ξ is a subordinator
2. Further studied by Carmona, Petit and Yor who have in special cases $\mathcal{M}_{I_\Psi}(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_{I_\Psi}(z)$
3. Maulik and Zwart derive this key recurrent equation in some generality and utilize it
4. Kuznetsov solves this recurrent relation for some classes of Lévy processes
5. There are various other contributions relying on different approaches

1. I_Ψ introduced and studied by Urbanik when ξ is a subordinator
2. Further studied by Carmona, Petit and Yor who have in special cases $\mathcal{M}_{I_\Psi}(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_{I_\Psi}(z)$
3. Maulik and Zwart derive this key recurrent equation in some generality and utilize it
4. Kuznetsov solves this recurrent relation for some classes of Lévy processes
5. There are various other contributions relying on different approaches

1. I_ψ introduced and studied by Urbanik when ξ is a subordinator
2. Further studied by Carmona, Petit and Yor who have in special cases $\mathcal{M}_{I_\psi}(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_{I_\psi}(z)$
3. Maulik and Zwart derive this key recurrent equation in some generality and utilize it
4. Kuznetsov solves this recurrent relation for some classes of Lévy processes
5. There are various other contributions relying on different approaches

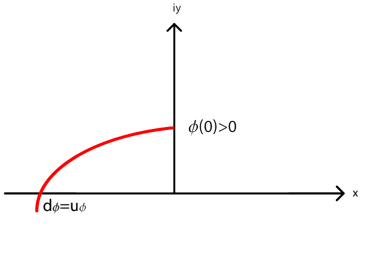
1. I_Ψ introduced and studied by Urbanik when ξ is a subordinator
2. Further studied by Carmona, Petit and Yor who have in special cases $\mathcal{M}_{I_\Psi}(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_{I_\Psi}(z)$
3. Maulik and Zwart derive this key recurrent equation in some generality and utilize it
4. Kuznetsov solves this recurrent relation for some classes of Lévy processes
5. There are various other contributions relying on different approaches

Key quantities of $\phi \in \mathcal{B}$ in relation to $W_\phi(z+1) = \phi(z)W_\phi(z)$

We use $A_{(a,b)}$ (resp. $M_{(a,b)}$) to denote the holomorphic (resp. meromorphic) functions on the complex strip $\mathbb{C}_{(a,b)} = \{z \in \mathbb{C} : \operatorname{Re}(z) \in (a,b)\}$.

We use $A_{(a,b)}$ (resp. $M_{(a,b)}$) to denote the holomorphic (resp. meromorphic) functions on the complex strip $\mathbb{C}_{(a,b)} = \{z \in \mathbb{C} : \operatorname{Re}(z) \in (a,b)\}$.
For any $\phi \in \mathcal{B}$ set

$$a_\phi = \inf_{u<0} \{\phi \in A_{(u,\infty)}\} \in [-\infty, 0]$$
$$u_\phi = \sup_{u\leq 0} \{\phi(u) = 0\} \in [-\infty, 0]$$
$$d_\phi = \sup_{u\leq 0} \{\phi(u) = 0 \text{ or } \phi(u) = -\infty\} \in [a_\phi, 0] .$$



Theorem

For any $\phi \in \mathcal{B}$

$$W_\phi(z) = \frac{1}{\phi(z)} e^{-\gamma_\phi z} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z} \in A_{(d_\phi,\infty)} \cap M_{(a_\phi,\infty)},$$

is a solution to $f(z+1) = \phi(z)f(z)$, $f(1) = 1$.

Theorem

For any $\phi \in \mathcal{B}$

$$W_\phi(z) = \frac{1}{\phi(z)} e^{-\gamma_\phi z} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z} \in A_{(d_\phi,\infty)} \cap M_{(a_\phi,\infty)},$$

is a solution to $f(z+1) = \phi(z)f(z)$, $f(1) = 1$. Moreover, W_ϕ is zero-free on $\mathbb{C}_{(a_\phi,\infty)}$ and $W_\phi(z+1) = \mathbb{E}\left[Y_\phi^z\right]$ for some positive random variable Y_ϕ.

Theorem

For any $\phi \in \mathcal{B}$

$$W_\phi(z) = \frac{1}{\phi(z)} e^{-\gamma_\phi z} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z} \in A_{(d_\phi,\infty)} \cap M_{(a_\phi,\infty)},$$

is a solution to $f(z+1) = \phi(z)f(z)$, $f(1) = 1$. Moreover, W_ϕ is zero-free on $\mathbb{C}_{(a_\phi,\infty)}$ and $W_\phi(z+1) = \mathbb{E}\left[Y_\phi^z\right]$ for some positive random variable Y_ϕ.

When $\phi(z) = z$, $d_\phi = 0$, $a_\phi = -\infty$, $W_\phi(z) = \Gamma(z)$.

Theorem

If $a, b > 0$, $z = a + ib$. Then

$$|W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a) - A_\phi(z)} \underbrace{e^{-E_\phi(z) - R_\phi(a)}}_{\text{error term}},$$

Theorem

If $a, b > 0$, $z = a + ib$. Then

$$|W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a) - A_\phi(z)} \underbrace{e^{-E_\phi(z) - R_\phi(a)}}_{\text{error term}},$$

where

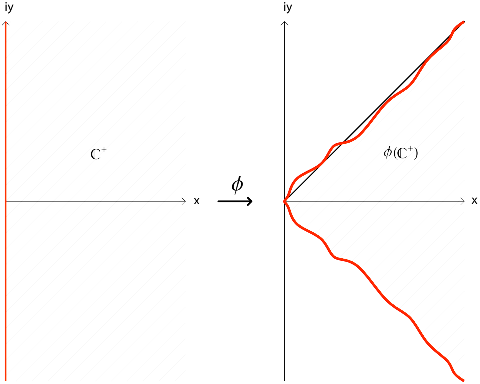
$$A_\phi(z) = \int_0^b \arg(\phi(a + iu))\, du,$$

$$\Theta_\phi(z) = \int_{\frac{a}{b}}^\infty \ln\left(\frac{|\phi(bu + ib)|}{\phi(bu)}\right) du = \frac{1}{b}\int_a^\infty \ln\left(\frac{|\phi(u + ib)|}{\phi(u)}\right) du$$

$$G_\phi(z) = G_\phi(a) = \int_1^{1+a} \ln\phi(u) du$$

and $\Theta_\phi(a + ib) = \frac{1}{b} A_\phi(a + ib) \in \left[0, \frac{\pi}{2}\right]$.

$$A_\phi(z) = \left(\frac{1}{b}\int_0^b \arg\phi(iu)\,du\right) \times b$$



$$|W_\phi(z)| \asymp e^{G_\phi(a) - A_\phi(z)}$$

1. If $a \to \infty$

$$G_\phi(a) \stackrel{\infty}{\approx} a \ln \phi(a) + \ln \phi(a) - (a+1)O(1)$$

2. For $\mathcal{B}_P = \{\phi \in \mathcal{B} : \delta > 0\}$ then the asymptotic along $a + i\mathbb{R}$ is

$$A_\phi(a+ib) \stackrel{\infty}{\approx} \frac{\pi}{2}|b| - \left(a + \frac{m}{\delta}\right) \ln |b| + o(|b|)$$

3. For $\mathcal{B}_\alpha = \left\{\phi \in \mathcal{B} : \delta = 0; \mu(dy) \stackrel{0}{\sim} y^{-\alpha-1}dy, \alpha\,(0,1)\right\}$

$$A_\phi(a+ib) \stackrel{\infty}{\approx} \frac{\pi}{2}\alpha|b| + o(|b|)$$

1. If $a \to \infty$

$$G_\phi(a) \overset{\infty}{\approx} a \ln \phi(a) + \ln \phi(a) - (a+1)O(1)$$

2. For $\mathcal{B}_P = \{\phi \in \mathcal{B} :\ \delta > 0\}$ then the asymptotic along $a + i\mathbb{R}$ is

$$A_\phi(a+ib) \overset{\infty}{\approx} \frac{\pi}{2}|b| - \left(a + \frac{m}{\delta}\right) \ln |b| + o(|b|)$$

3. For $\mathcal{B}_\alpha = \left\{\phi \in \mathcal{B} :\ \delta = 0; \mu(dy) \overset{0}{\sim} y^{-\alpha-1}dy,\ \alpha\,(0,1)\right\}$

$$A_\phi(a+ib) \overset{\infty}{\approx} \frac{\pi}{2}\alpha|b| + o(|b|)$$

Discussion $|W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a)-A_\phi(z)} e^{-E_\phi(z)-R_\phi(a)}$

1. If $a \to \infty$

$$G_\phi(a) \overset{\infty}{\approx} a\ln\phi(a) + \ln\phi(a) - (a+1)O(1)$$

2. For $\mathcal{B}_P = \{\phi \in \mathcal{B} : \delta > 0\}$ then the asymptotic along $a + i\mathbb{R}$ is

$$A_\phi(a+ib) \overset{\infty}{\approx} \frac{\pi}{2}|b| - \left(a + \frac{m}{\delta}\right)\ln|b| + o(|b|)$$

3. For $\mathcal{B}_\alpha = \left\{\phi \in \mathcal{B} : \delta = 0; \mu(dy) \overset{0}{\sim} y^{-\alpha-1}dy,\ \alpha\,(0,1)\right\}$

$$A_\phi(a+ib) \overset{\infty}{\approx} \frac{\pi}{2}\alpha|b| + o(|b|)$$

The celebrated factorization Wiener-Hopf

$$\Psi(-z) = -\phi_+(z)\phi_-(-z), \text{ at least for } z \in i\mathbb{R}$$

with

$$\phi_\pm(z) = m_\pm + \delta_\pm z + \int_0^\infty \left(1 - e^{-zy}\right) \mu_\pm(dy),\ z \in \mathbb{C}_{(0,\infty)},$$

are Bernstein functions then yields

$$f(z+1) = -\frac{-z}{\Psi(-z)} f(z) = \frac{z}{\phi_+(z)} \frac{1}{\phi_-(-z)} f(z), \tag{0.3}$$

on $\{z \in i\mathbb{R} : \Psi(-z) \neq 0\}$.

The product of the solutions to the independent system

$$\begin{aligned} f_1(z+1) &= \frac{z}{\phi_+(z)} f_1(z) \\ f_2(z+1) &= \frac{1}{\phi_-(-z)} f_2(z) \end{aligned}$$

on a <u>common</u> complex domain is a solution to $f(z+1) = \frac{-z}{\Psi(-z)} f(z)$ on this domain.

The product of the solutions to the independent system

$$\begin{aligned} f_1(z+1) &= \frac{z}{\phi_+(z)} f_1(z) \\ f_2(z+1) &= \frac{1}{\phi_-(-z)} f_2(z) \end{aligned}$$

on a common complex domain is a solution to $f(z+1) = \frac{-z}{\Psi(-z)} f(z)$ on this domain.

These can be extracted from the general solution to $f_\pm(z+1) = \phi_\pm(z) f_\pm(z)$ that is $W_{\phi_\pm}$.

Theorem

Let $\Psi \in \overline{\mathcal{N}}$. Then

$$\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) \in A_{\left(a_{\phi_+} 1_{\left\{d_{\phi_+}=0\right\}}, 1-d_{\phi_-}\right)} \cap M_{\left(a_{\phi_+}, 1-a_{\phi_-}\right)}$$

solves $f(z+1) = \frac{-z}{\Psi(-z)} f(z)$.

Theorem

Let $\Psi \in \overline{\mathcal{N}}$. Then

$$\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) \in A_{\left(a_{\phi_+} 1_{\left\{d_{\phi_+}=0\right\}}, 1-d_{\phi_-}\right)} \cap M_{\left(a_{\phi_+}, 1-a_{\phi_-}\right)}$$

solves $f(z+1) = \frac{-z}{\Psi(-z)} f(z)$. Also if $\Psi \in \mathcal{N}$ then

$$\mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} \phi_-(0) W_{\phi_-}(1-z).$$

Theorem

Let $\Psi \in \overline{\mathcal{N}}$. Then

$$\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) \in A_{\left(a_{\phi_+} 1_{\left\{d_{\phi_+}=0\right\}}, 1-d_{\phi_-}\right)} \cap M_{\left(a_{\phi_+}, 1-a_{\phi_-}\right)}$$

solves $f(z+1) = \frac{-z}{\Psi(-z)} f(z)$. Also if $\Psi \in \mathcal{N}$ then

$$\mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} \phi_-(0) W_{\phi_-}(1-z).$$

As a consequence of the Weierstrass product representations of $W_{\phi_\pm}, \Gamma$

$$I_\Psi \stackrel{d}{=} I_{\phi_+} \times X_{\phi_-} \stackrel{d}{=} \bigotimes_{k=0}^{\infty} C_k Y_k,$$

where $\mathbb{E}\left[f\left(Y_k\right)\right] = \frac{\mathbb{E}\left[Y_0^k f(Y_0)\right]}{\mathbb{E}\left[Y_0^k\right]}$.

Theorem

Let $\Psi \in \overline{\mathcal{N}}$. Then exists $N_\Psi \in (0,\infty]$ such that for any $a \in \left(0, 1 - d_{\phi_-}\right)$

$$\lim_{|b|\to\infty} |b|^\eta \left|\mathcal{M}_\Psi(a+ib)\right| = 0 \iff \eta \in (0, N_\Psi).$$

Therefore if $\Psi \in \mathcal{N}$, $p_\Psi \in C_0^{\lfloor N_\Psi \rfloor - 1}(\mathbb{R}^+)$ if $N_\Psi > 1$.

Theorem

Let $\Psi \in \overline{\mathcal{N}}$. Then exists $N_\Psi \in (0,\infty]$ such that for any $a \in \left(0, 1-d_{\phi_-}\right)$

$$\lim_{|b|\to\infty} |b|^\eta \left|\mathcal{M}_\Psi\left(a+ib\right)\right| = 0 \iff \eta \in \left(0, N_\Psi\right).$$

Therefore if $\Psi \in \mathcal{N}$, $p_\Psi \in C_0^{\lfloor N_\Psi \rfloor - 1}\left(\mathbb{R}^+\right)$ if $N_\Psi > 1$.

$$N_\Psi = \frac{m_-(0)}{\bar{\mu}_-(0) + \phi_-(0)} + \frac{\phi_+(0) + \bar{\mu}_+(0)}{\delta_+} \in (0,\infty)$$

if and only if Ψ corresponds to $\xi_t = \delta_+ t + \sum_{j=1}^{N_t} X_j$, $\delta_+ > 0$.

Theorem

Let $\Psi \in \overline{\mathcal{N}}$. Then exists $N_\Psi \in (0,\infty]$ such that for any $a \in \left(0, 1-d_{\phi_-}\right)$

$$\lim_{|b|\to\infty} |b|^\eta \left|\mathcal{M}_\Psi\left(a+ib\right)\right| = 0 \iff \eta \in \left(0, N_\Psi\right).$$

Therefore if $\Psi \in \mathcal{N}$, $p_\Psi \in C_0^{\lfloor N_\Psi \rfloor - 1}\left(\mathbb{R}^+\right)$ if $N_\Psi > 1$.

$$N_\Psi = \frac{m_-(0)}{\bar{\mu}_-(0) + \phi_-(0)} + \frac{\phi_+(0) + \bar{\mu}_+(0)}{\delta_+} \in (0,\infty)$$

if and only if Ψ corresponds to $\xi_t = \delta_+ t + \sum_{j=1}^{N_t} X_j$, $\delta_+ > 0$.

N_Ψ is a measure for the polynomial decay of $|\mathcal{M}_\Psi|$ along complex lines.

1. For fixed $a \in \left(0, 1 - d_{\phi_-}\right)$

$$
\begin{aligned}
|\mathcal{M}_\Psi(z)| &= \left|\frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z)\right| \\
&= C\left|\frac{\sqrt{\phi_+(z)}}{\sqrt{\phi_-(z)}}\Gamma(z)\right| e^{-A_{\phi_-}(1-z)+A_{\phi_+}(z)} \\
&\overset{\infty}{\approx} C|b|^{a-\frac{1}{2}} e^{-\frac{\pi}{2}|b|-A_{\phi_-}(1-a-ib)+A_{\phi_+}(a+ib)}
\end{aligned}
$$

2. The hardest case is when $\delta_+ > 0$, $\delta_- = 0$. Depending on $\overline{\Pi}_-(y)$ as $y \to 0$ we use different techniques-reducing to $\overline{\Pi}_+(0) = 0$, using the alternative representation Θ_ϕ for A_ϕ, etc.

1. For fixed $a \in \left(0, 1 - d_{\phi_-}\right)$

$$\begin{aligned} |\mathcal{M}_\Psi(z)| &= \left|\frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z)\right| \\ &= C \left|\frac{\sqrt{\phi_+(z)}}{\sqrt{\phi_-(z)}}\Gamma(z)\right| e^{-A_{\phi_-}(1-z)+A_{\phi_+}(z)} \\ &\overset{\infty}{\approx} C|b|^{a-\frac{1}{2}} e^{-\frac{\pi}{2}|b|-A_{\phi_-}(1-a-ib)+A_{\phi_+}(a+ib)} \end{aligned}$$

2. The hardest case is when $\delta_+ > 0$, $\delta_- = 0$. Depending on $\overline{\Pi}_-(y)$ as $y \to 0$ we use different techniques-reducing to $\overline{\Pi}_+(0) = 0$, using the alternative representation Θ_ϕ for A_ϕ, etc.

Theorem

Let $\Psi \in \mathcal{N}$ that is $I_\Psi < \infty$. Then

$$\lim_{x\to\infty} \frac{\ln \mathbb{P}(I_\Psi > x)}{\ln(x)} = d_{\phi_-} = \sup_{u\leq 0} \{\phi_-(u) = 0 \text{ or } \phi_-(u) = -\infty\} \in [-\infty, 0], \tag{0.4}$$

where recall that $\Psi(z) = -\phi_+(-z)\phi_-(z)$.

Theorem

Let $\Psi \in \mathcal{N}$ that is $I_\Psi < \infty$. Then

$$\lim_{x\to\infty} \frac{\ln \mathbb{P}(I_\Psi > x)}{\ln(x)} = d_{\phi_-} = \sup_{u\leq 0} \{\phi_-(u) = 0 \text{ or } \phi_-(u) = -\infty\} \in [-\infty, 0], \tag{0.4}$$

where recall that $\Psi(z) = -\phi_+(-z)\phi_-(z)$.

If $\exists\, \theta_\Psi < 0 : \Psi(\theta_\Psi) = 0$ and $\left|\Psi\left(\theta_\Psi^+\right)\right| < \infty$ then

$$\lim_{x\to\infty} x^{-\theta_\Psi+n+1} p_\Psi^{(n)}(x) = C > 0 \tag{0.5}$$

provided $N_\Psi > n+1$ and a weak non-lattice condition when $n \geq 1$.

Small behaviour of I_Ψ: the role of $\mathcal{M}_{I_\Psi}(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)}\phi_-(0)W_{\phi_-}(1-z)$

The small behaviour depends on the poles of $\frac{\Gamma(z)}{W_{\phi_+}(z)}$ on $\mathbb{C}_{(a_{\phi_+},1)}$.

- If $\phi_+(0)=0$ then $\frac{\Gamma(z)}{W_{\phi_+}(z)} \in A_{(a_{\phi_+},\infty)}$ and then $\mathbb{P}(I_\Psi \le x) = o(x^{-a}), \forall a \in (a_{\phi_+},\infty)$ and $\mathbb{P}(I_\Psi \le x) = o(1)$ if $a_{\phi_+}=0$
- If $\phi_+(0)>0$, then $\frac{\Gamma(z)}{W_{\phi_+}(z)} \in M_{(a_{\phi_+},\infty)}$ and $\Psi(0)=-q$ with

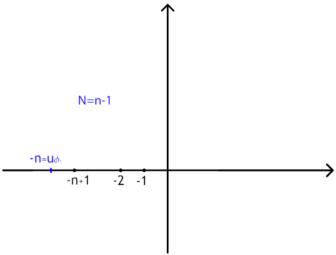
$$\mathbb{P}(I_\Psi \le x) = q\sum_{j=1}^{N} \frac{\prod_{k=1}^{j-1}\Psi(k)}{j!}x^j - \frac{\phi_-(0)}{2\pi i}\int_{a-i\infty}^{a+i\infty} x^{-z}\mathcal{M}_\Psi(z)dz,$$

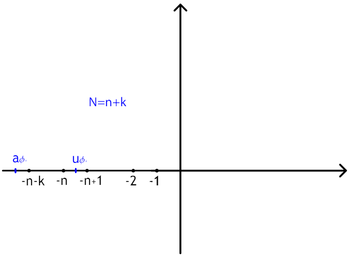
Small behaviour of I_Ψ: the role of $\mathcal{M}_{I_\Psi}(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)}\phi_-(0)W_{\phi_-}(1-z)$

The small behaviour depends on the poles of $\frac{\Gamma(z)}{W_{\phi_+}(z)}$ on $\mathbb{C}_{\left(a_{\phi_+},1\right)}$.

- If $\phi_+(0) = 0$ then $\frac{\Gamma(z)}{W_{\phi_+}(z)} \in A_{\left(a_{\phi_+},\infty\right)}$ and then $\mathbb{P}\left(I_\Psi \leq x\right) = o\left(x^{-a}\right)$, $\forall a \in \left(a_{\phi_+},\infty\right)$ and $\mathbb{P}\left(I_\Psi \leq x\right) = o\left(1\right)$ if $a_{\phi_+} = 0$
- If $\phi_+(0) > 0$, then $\frac{\Gamma(z)}{W_{\phi_+}(z)} \in M_{\left(a_{\phi_+},\infty\right)}$ and $\Psi(0) = -q$ with

$$\mathbb{P}\left(I_\Psi \leq x\right) = q\sum_{j=1}^{N}\frac{\prod_{k=1}^{j-1}\Psi(k)}{j!}x^j - \frac{\phi_-(0)}{2\pi i}\int_{a-i\infty}^{a+i\infty}x^{-z}\mathcal{M}_\Psi(z)dz,$$





Let $\Psi \in \overline{\mathcal{N}}$ and set $I_\Psi(t) = \int_0^t e^{-\xi_s} ds$. Clearly with $\Psi_q(z) = \Psi(z) - q = -\phi_+^q(-z)\phi_-^q(z) \in \mathcal{N}$ we have with real a that

$$\frac{1}{q}\mathcal{M}_{I_{\Psi_q}}(a) = \frac{1}{q}\mathbb{E}\left[I_{\Psi_q}^{a-1}\right] = \underbrace{\int_0^\infty e^{-qt}\mathbb{E}\left[I_\Psi^{a-1}(t)\right] dt}_{\text{Laplace transform}}$$

$$= \frac{\Gamma(a)}{W_{\phi_+^q}(a)} \frac{\phi_-^q(0)}{q} W_{\phi_-^q}(1-a)$$

Theorem

Let $\Psi \notin \mathcal{N}$ with $\limsup_{t\to\infty} \xi_t = \limsup_{t\to\infty} -\xi_t = \infty$ and $\lim_{t\to\infty} \mathbb{P}(\xi_t < 0) = \rho \in [0,1)$. Set $\Psi_r(\cdot) = \Psi(\cdot) - r = -\phi_+^r(-\cdot)\phi_-^r(\cdot) \in \mathcal{N}$ and $\kappa_-(r) = \phi_-^r(0)$. Then $\kappa_- \in \mathrm{RV}(\rho)$ at zero and for any $a \in (0,1)$, $f \in C_b(\mathbb{R}^+)$

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[\mathrm{I}_\Psi^{-a}(t) f(\mathrm{I}_\Psi(t))\right]}{\kappa_-\left(\frac{1}{t}\right)} = \int_0^\infty f(x)\vartheta_a(dx).$$

Theorem

Let $\Psi\notin\mathcal{N}$ with $\limsup_{t\to\infty}\xi_t=\limsup_{t\to\infty}-\xi_t=\infty$ and $\lim_{t\to\infty}\mathbb{P}(\xi_t<0)=\rho\in[0,1)$. Set $\Psi_r(\cdot)=\Psi(\cdot)-r=-\phi_+^r(-\cdot)\phi_-^r(\cdot)\in\mathcal{N}$ and $\kappa_-(r)=\phi_-^r(0)$. Then $\kappa_-\in RV(\rho)$ at zero and for any $a\in(0,1)$, $f\in C_b(\mathbb{R}^+)$

$$\lim_{t\to\infty}\frac{\mathbb{E}\left[I_\Psi^{-a}(t)f\left(I_\Psi(t)\right)\right]}{\kappa_-\left(\frac{1}{t}\right)}=\int_0^\infty f(x)\vartheta_a(dx).$$

If $\mathbb{E}[\xi_1]=0,\mathbb{E}\left[\xi_1^2\right]<\infty$ then $\kappa_-(r)\overset{0}{\sim}Cr^{\frac{1}{2}}$.

Thank you!