Markov Chain Approximation of Pure Jump Processes

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Outline

Problem

Let $\{\mathbf{X}^n\}_{n\in\mathbb{N}}$ be a sequence of Markov chains on $\mathbb{Z}_n^d := n^{-1}\mathbb{Z}^d$, and let $\mathbf{X}$ be a Markov process on $\mathbb{R}^d$.

Problem

Let $\{\mathbf{X}^n\}_{n\in\mathbb{N}}$ be a sequence of Markov chains on $\mathbb{Z}^d_n := n^{-1}\mathbb{Z}^d$, and let $\mathbf{X}$ be a Markov process on $\mathbb{R}^d$. The following two questions naturally arise:

- When does $\{\mathbf{X}^n\}_{n\in\mathbb{N}}$ converge weakly to a Markov process?
- Can $\mathbf{X}$ be approximated (in the sense of weak convergence) by a sequence of Markov chains?

Related results

- Stroock-Varadhan, *Multidimensional diffusion processes* 1979: **X** is a diffusion process determined by a generator in non-divergence form.
- Stroock-Zheng, AIHP 1997: **X** is a symmetric diffusion process determined by a generator in divergence form.
- Bass-Kumagai, TAMS 2008: **X** is a symmetric diffusion process determined by a generator in divergence form.
- Deuschel-Kumagai, CPAM 2013: **X** is a non-symmetric diffusion process determined by a generator in divergence form.

Related results

- Husseini-Kassmann, PA 2007: **X** is a symmetric pure jump process whose corresponding jump kernel is comparable to the jump kernel of a symmetric stable Lévy process.
- Bass-Kassmann-Kumagai, AIHP 2010: **X** is a symmetric pure jump process with “stable-like” kernel.
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The main step in the proofs of all above mentioned results is to obtain a prior heat kernel estimates of the chains $\{\mathbf{X}^n\}_{n\in\mathbb{N}}$.

Related results

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Their approach consists of two steps:

- to conclude tightness of $\{\mathbf{X}^n\}_{n\in\mathbb{N}}$ they use the Lyons-Zhang decomposition, Lyons-Zhang, AP 1994;
- to prove convergence of finite-dimensional distributions of $\{\mathbf{X}^n\}_{n\in\mathbb{N}}$ to finite-dimensional distributions of $\mathbf{X}$ they apply the Mosco convergence of symmetric Dirichlet forms, obtained by Mosco, JFA 1994, and generalized by Kim, SPA 2006.

Goal

Discuss the questions of convergence and approximation in the case when $\mathbf{X}$ is a non-symmetric pure jump process.

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The approach consists of two steps:

- to conclude tightness of $\{\mathbf{X}^n\}_{n\in\mathbb{N}}$ we use stochastic analysis tools (characteristics of semimartingales) discussed in Jacod-Shiryaev, *Limit theorems for stochastic processes*, 2003;
- to prove convergence of finite-dimensional distributions of $\{\mathbf{X}^n\}_{n\in\mathbb{N}}$ to finite-dimensional distributions of $\mathbf{X}$ we apply the Mosco convergence of non-symmetric Dirichlet forms, obtained by Hino, JMKU 1998, and generalized by Tölle, Master's thesis 2006.

Semimartingale approach

Let $\{S_t\}_{t\geq 0}$ a semimatingale and let $h : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a truncation function. Define,

$$\bar{S}(h)_t := \sum_{s\leq t}(\Delta S_s - h(\Delta S_s)) \quad \text{and} \quad S(h)_t := S_t - \bar{S}(h)_t.$$

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The process $\{S(h)_t\}_{t\geq 0}$ is a special semimartingale, that is, it admits a unique decomposition

$$S(h)_t = S_0 + M(h)_t + B(h)_t,$$

where $\{M(h)_t\}_{t\geq 0}$ is a local martingale and $\{B(h)_t\}_{t\geq 0}$ is a predictable process of bounded variation.

Semimartingale approach

Further, let $N(\omega, ds, dy)$ be the compensator of the jump measure

$$\mu(\omega, ds, dy) := \sum_{s:\Delta S_s(\omega)\neq 0} \delta_{(s,\Delta S_s(\omega))}(ds, dy)$$

of $\{S_t\}_{t\geq 0}$, and let $\{A_t\}_{t\geq 0} = \{(A_t^{ij})_{1\leq i,j\leq d})\}_{t\geq 0}$ be the quadratic co-variation process for $\{S_t^c\}_{t\geq 0}$.

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In addition, by defining $\tilde{A}(h)_t^{ij} := \langle M(h)_t^i, M(h)_t^j\rangle$, the triplet $(B, \tilde{A}, N)$ is called the modified characteristics of $\{S_t\}_{t\geq 0}$ (relative to $h(x)$).

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Jacod-Shiryaev, *Limit theorems for stochastic processes*, 2003: Problem of weak convergence of a sequence of semimartingales to a semimartingale translate in terms of convergence of the corresponding characteristics.

Semimartingale approach

Let $C^n : \mathbb{Z}_n^d \times \mathbb{Z}_n^d \longrightarrow [0,\infty)$, $n \in \mathbb{N}$, be a family of functions satisfying

(S1) $C^n(a,a) = 0$ for all $a \in \mathbb{Z}_n^d$ and all $n \in \mathbb{N}$;

(S2) $\sup_{a\in\mathbb{Z}_n^d} \sum_{b\in\mathbb{Z}_n^d} C^n(a,b) < \infty$ for all $n \in \mathbb{N}$.

Semimartingale approach

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Then, under (**S1**) and (**S2**), C^n, $n \in \mathbb{N}$, define a family of regular continuous-time Markov chains $\{X_t^n\}_{t\geq 0}$ on $\mathbb{Z}_n^d$ determined by infinitesimal generator of the form

$$\mathcal{A}^n f(a) = \sum_{b\in\mathbb{Z}_n^d} (f(b) - f(a))C^n(a, b).$$

Semimartingale approach

Clearly, the processes $\{X_t^n\}_{t\geq 0}$, $n \in \mathbb{N}$, are semimartingales and the corresponding (modified) characteristics are given by:

$$\begin{aligned}
B^n(h)_t &= \int_0^t \sum_{b\in\mathbb{Z}_n^d} h(b)C^n(X_s^n, X_s^n + b)ds,\\
A_t^n &= 0,\\
\tilde{A}^n(h)_t^{ij} &= \int_0^t \sum_{b\in\mathbb{Z}_n^d} h_i(b)h_j(b)C^n(X_s^n, X_s^n + b)ds,\\
N^n(ds, b) &= C^n(X_s^n, X_s^n + b)ds.
\end{aligned}$$

Semimartingale approach

Pure jump homogeneous diffusion with jumps is a semimartingale $\{X_t\}_{t\geq 0}$ determined with (modified) characteristics of the form

$$\begin{aligned} B_t &:= \int_0^t b(X_s)ds, \\ A_t^{n,ij} &:= 0, \quad i,j = 1,\dots,d, \\ \tilde{A}_t^{n,ij} &:= \int_0^t \int_{\mathbb{R}^d} h_i(y)h_j(y)\nu(X_s,dy)ds, \quad i,j = 1,\dots,d, \\ N(ds,dy) &:= \nu(X_s,dy)ds, \end{aligned}$$

where $b : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\nu : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \longrightarrow [0,\infty]$ are, respectively, Borel function and Borel kernel satisfying $\nu(x,\{0\}) = 0$ and

$$\int_{\mathbb{R}^d} (1 \wedge |y|^2)\nu(x,dy) < \infty.$$

Semimartingale approach

For $a = (a_1, \dots, a_d) \in \mathbb{Z}_n^d$ set

$$\bar{a} := [a_1 - 1/2n, a_1 + 1/2n) \times \cdots \times [a_d - 1/2n, a_d + 1/2n),$$

and for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ define

$$[x]_n := \left([nx_1 + 1/2n] / n, \dots, [nx_d + 1/2n] / n \right).$$

Note that for $a \in \mathbb{Z}_n^d$, $[x]_n = a$ for all $x \in \bar{a}$.

Semimartingale approach

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Note that for $a\in\mathbb{Z}_n^d$, $[x]_n=a$ for all $x\in\bar{a}$.

Theorem

Under **(S1)**, **(S2)**,

(S3) the functions $b(x)$, $x\longmapsto\int_{\mathbb{R}^d}h_i(y)h_j(y)\nu(x,dy)$ and $x\longmapsto\int_{\mathbb{R}^d}g(y)\nu(x,dy)$ are continuous for any bounded and continuous function $g:\mathbb{R}^d\longrightarrow\mathbb{R}$ vanishing in a neighborhood of the origin;

(S4) for all $R>0$,

$$\lim_{r\nearrow\infty}\sup_{x\in B_R(0)}\nu(x,B_r^c(0))=0;$$

Semimartingale approach

Theorem (continued)

(S5) for all $R > 0$,

$$\lim_{n\nearrow\infty} \sup_{x\in B_R(0)} \left| \sum_{b\in\mathbb{Z}_n^d} h_i(b) C^n([x]_n, [x]_n + b) - b_i(x) \right| = 0;$$

(S6) for all $R > 0$,

$$\lim_{n\nearrow\infty} \sup_{x\in B_R(0)} \left| \sum_{b\in\mathbb{Z}_n^d} h_i(b) h_j(b) C^n([x]_n, [x]_n + b) - \int_{\mathbb{R}^d} h_i(y) h_j(y) \nu(x, dy) \right| = 0;$$

Semimartingale approach

Theorem (continued)

(S7) for all $R > 0$ and all bounded and continuous functions $g : \mathbb{R}^d \longrightarrow \mathbb{R}$ vanishing in a neighbourhood of the origin,

$$\lim_{n \nearrow \infty} \sup_{x \in B_R(0)} \left| \sum_{b \in \mathbb{Z}^d_n} g(b) C^n([x]_n, [x]_n + b) - \int_{\mathbb{R}^d} g(y) \nu(x, dy) \right| = 0,$$

$$\{X^n_t\}_{t \geq 0} \xrightarrow[n \nearrow \infty]{d} \{X_t\}_{t \geq 0}.$$

Semimartingale approach

Assume

$$\sup_{x\in\mathbb{R}^d} \nu(x, B_\rho^c(0)) < \infty, \quad \rho > 0.$$

For $0 < p \leq 1$, define $C^{n,p} : \mathbb{Z}_n^d \times \mathbb{Z}_n^d \longrightarrow [0, \infty)$ by

$$C^{n,p}(a, b) := \begin{cases} \nu(a, \bar{b} - a), & |a - b| > \frac{\sqrt{d}}{n^p} \\ 0, & |a - b| \leq \frac{\sqrt{d}}{n^p}. \end{cases}$$

Observe that $C^{n,p}$, $n \in \mathbb{N}$, automatically satisfy (**S1**) and (**S2**).

Semimartingale approach

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Observe that $C^{n,p}$, $n \in \mathbb{N}$, automatically satisfy (**S1**) and (**S2**).

Theorem

The conditions in (**S5**)-(**S7**) will be satisfied if for all $\rho > 0$ and $R > 0$,

- $\lim_{\varepsilon\searrow 0} \varepsilon \sup_{x\in B_R(0)} \nu(x, B_\rho(0) \setminus B_{\varepsilon^p}(0)) = 0,$
- $\lim_{\varepsilon\searrow 0} \varepsilon^p \sup_{x\in B_R(0)} \nu(x, B_{\sqrt{d}\varepsilon^p+(\sqrt{d}/2)\varepsilon}(0) \setminus B_{\sqrt{d}\varepsilon^p-(\sqrt{d}/2)\varepsilon}(0)) = 0,$

Semimartingale approach

Theorem (continued)

- $$\lim_{\varepsilon\searrow 0}\sup_{x\in B_R(0)}\int_{B_\varepsilon(0)}|y|^2\nu(x,dy)=0,$$
- $$\lim_{n\nearrow\infty}\sup_{x\in B_R(0)}\int_{B_\rho(0)}|y|^2\|\nu([x]_n,dy)-\nu(x,dy)\|_{TV}=0,$$
- $$\lim_{n\nearrow\infty}\sup_{x\in B_R(0)}\int_{B_\rho(0)\setminus B_{\sqrt{d}/n^p-\sqrt{d}/2n}(0)}|y|\|\nu([x]_n,dy)-\nu(x,dy)\|_{TV}=0,$$
- $$\lim_{n\nearrow\infty}\sup_{x\in B_R(0)}\|\nu([x]_n,B^c_\varepsilon(0))-\nu(x,B^c_\varepsilon(0))\|_{TV}=0,\quad \varepsilon>0,$$
- $$\lim_{\varepsilon\searrow 0}\sup_{x\in B_R(0)}\left|\int_{B^c_\varepsilon(0)}h_i(y)\nu(x,dy)-b_i(x)\right|=0.$$

Semimartingale approach

Examples

- Pure jump Lévy processes.

Semimartingale approach

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- Stable-like processes: Let $\alpha : \mathbb{R}^d \longrightarrow (0,2)$ be bounded and continuously differentiable function with bounded derivatives such that $0 < \underline{\alpha} = \alpha(x) = \overline{\alpha} < 2$. Under this assumptions, Bass, PTRF 1988, Schilling, PTRF 1998, and Schilling-Wang, TAMS 2013, have shown that there exists a unique Feller semimartingale $\{X_t\}_{t\geq 0}$, called a stable-like process, determined by (modified) characteristics (with respect to an odd truncation function $h(x)$) of the form

$$B(h)_t = 0,$$

$$\tilde{A}_t^{i,j} = \int_0^t \int_{\mathbb{R}^d} h_i(y)h_j(y)\frac{dy}{|y|^{d+\alpha(X_s)}}ds,$$

$$N(ds, dy) = \frac{dyds}{|y|^{d+\alpha(X_s)}}.$$

Semimartingale approach

Examples

- Lévy-driven SDEs: Let $\{L_t\}_{t\geq 0}$ be an n-dimensional Lévy process and let $\Phi : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times n}$ be bounded and locally Lipschitz continuous. Then, Schilling-Schnurr, EJP 2010, have shown that the SDE

$$dX_t = \Phi(X_{t-})dL_t, \quad X_0 = x \in \mathbb{R}^d,$$

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Semimartingale approach

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admits a unique strong solution which is a Feller semimartingale.

In particular, if

- $L_t = (l_t, t)$, where $\{l_t\}_{t\geq 0}$ is a d-dimensional Lévy process determined by Lévy triplet $(0, 0, \nu(dy))$ such that $\nu(dy)$ is symmetric;
- $\Phi(x) = (\phi(x)I, 0)$, where $\phi : \mathbb{R}^d \longrightarrow \mathbb{R}$ is locally Lipschitz continuous and $0 < \inf_{x\in\mathbb{R}^d} |\phi(x)| \leq \sup_{x\in\mathbb{R}^d} |\phi(x)| < \infty$,

Semimartingale approach

Examples

then $\{X_t\}_{t\geq 0}$ is determined by (modified) characteristics (with respect to an odd truncation function $h(x)$) of the form

$$B(h)_t = 0,$$
$$\tilde{A}^{i,j}_t = \int_0^t \int_{\mathbb{R}^d} h_i(y)h_j(y)\nu\left(dy/|\phi(X_s)|\right) ds,$$
$$N(ds, dy) = \nu\left(dy/|\phi(X_s)|\right).$$

Dirichlet form approach

Recall, functions $C^n : \mathbb{Z}_n^d \times \mathbb{Z}_n^d \longrightarrow [0,\infty)$, $n \in \mathbb{N}$, satisfying

(T1) $C^n(a,a) = 0$ for all $a \in \mathbb{Z}_n^d$ and all $n \in \mathbb{N}$;

(T2) $\sup_{a\in\mathbb{Z}_n^d} \sum_{b\in\mathbb{Z}_n^d} C^n(a,b) < \infty$ for all $n \in \mathbb{N}$,

define a family of regular continuous-time Markov chains $\{X_t^n\}_{t\geq 0}$ with (modified) characteristics:

$$B^n(h)_t = \int_0^t \sum_{b\in\mathbb{Z}_n^d} h(b)C^n(X_s^n, X_s^n + b)ds,$$

$$A_t^n = 0,$$

$$\tilde{A}^n(h)_t^{ij} = \int_0^t \sum_{b\in\mathbb{Z}_n^d} h_i(b)h_j(b)C^n(X_s^n, X_s^n + b)ds,$$

$$N^n(ds, b) = C^n(X_s^n, X_s^n + b)ds.$$

Dirichlet form approach

Theorem (tightness)

The family $\{X_t^n\}_{t\geq 0}$ will be tight if **(T1)**, **(T2)** and

(T3) $\limsup_{n\nearrow\infty} \sup_{a\in\mathbb{Z}_n^d} \sum_{|b|>\rho} C^n(a,a+b) < \infty$, $\rho > 0$,

$$\lim_{r\nearrow\infty} \limsup_{n\nearrow\infty} \sup_{a\in\mathbb{Z}_n^d} \sum_{|b|>r} C^n(a,a+b) = 0;$$

(T4) there exists $\rho > 0$ such that

$$\limsup_{n\nearrow\infty} \sup_{a\in\mathbb{Z}_n^d} \left| \sum_{|b|<\rho} b_i C^n(a,a+b) \right| < \infty$$

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hold true.

Dirichlet form approach

Let $k : \mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag} \longrightarrow [0, \infty)$ be a Borel measurable function. Denote

$$k_s(x, y) := \frac{1}{2}(k(x, y) + k(y, x))$$
$$k_a(x, y) := \frac{1}{2}(k(x, y) - k(y, x)).$$

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Under assumption

(C1)
$$x \longmapsto \int_{\mathbb{R}^d} (1 \wedge |y|^2) k_s(x, x + y) dy \in L^1_{loc}(\mathbb{R}^d, dx)$$

$$\alpha_0 := \sup_{x \in \mathbb{R}^d} \int_{\{y \in \mathbb{R}^d : \, k_s(x,y) \neq 0\}} \frac{k_a(x, y)^2}{k_s(x, y)} dy < \infty,$$

Dirichlet form approach

$k(x,y)$ defines a regular symmetric Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mathbb{R}^d,dx)$, where

$$\mathcal{E}(f,g) := \int_{\mathbb{R}^d\times\mathbb{R}^d\setminus\mathrm{diag}} (f(y)-f(x))(g(y)-g(x))k_s(x,y)dxdy, \quad f,g\in\bar{\mathcal{F}},$$
$$\bar{\mathcal{F}} := \{f\in L^2(\mathbb{R}^d,dx) : \mathcal{E}(f,f)<\infty\}$$

and $\mathcal{F}$ is the $\mathcal{E}_1^{1/2}$-closure of $C_c^{Lip}(\mathbb{R}^d)$ in $\bar{\mathcal{F}}$.

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and $\mathcal{F}$ is the $\mathcal{E}_1^{1/2}$-closure of $C_c^{Lip}(\mathbb{R}^d)$ in $\bar{\mathcal{F}}$.

Further, Fukushima-Uemura, AP 2012, and Schilling-Wang, FMF 2015, have shown that the (non-symmetric) form

$$H(f,g) := -\lim_{\varepsilon\searrow 0}\int_{\mathbb{R}^d}\int_{B^c_\varepsilon(x)}(f(y)-f(x))k(x,y)dy\, g(x)dx,\ f,g\in C_c^{Lip}(\mathbb{R}^d),$$

Dirichlet form approach

is well defined, has a representation

$$H(f,g) = \frac{1}{2}\mathcal{E}(f,g) - \int_{\mathbb{R}^d \times \mathbb{R}^d \setminus \mathrm{diag}} (f(y) - f(x))g(y)k_a(x,y)dxdy,$$

extends to $\mathcal{F} \times \mathcal{F}$ such that $(H, \mathcal{F})$ defines a regular lower bounded coercive semi-Dirichlet form on $L^2(\mathbb{R}^d, dx)$ (and hence a Hunt process $(\{X_t\}_{t\geq 0}, \{\mathbb{P}^x\}_{x\in\mathbb{R}^d})$ defined on the complement of an exceptional set).

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extends to $\mathcal{F}\times\mathcal{F}$ such that $(H,\mathcal{F})$ defines a regular lower bounded coercive semi-Dirichlet form on $L^2(\mathbb{R}^d,dx)$ (and hence a Hunt process $(\{X_t\}_{t\geq 0},\{\mathbb{P}^x\}_{x\in\mathbb{R}^d})$ defined on the complement of an exceptional set). Moreover, it satisfies

$$\frac{1}{4}(1\wedge\alpha_0)\mathcal{E}_1(f)\leq H_{\alpha_0}(f)\leq\frac{2+\sqrt{2}}{2}(1\vee\alpha_0)\,\mathcal{E}_1(f),\quad f\in\mathcal{F},$$

and

$$(1\wedge\alpha_0)H_1(f)\leq H_{\alpha_0}(f)\leq(1\vee\alpha_0)\,H_1(f),\quad f\in\mathcal{F}.$$

Dirichlet form approach

Denote by $L^2(\mathbb{Z}_n^d)$ the standard Hilbert space on $\mathbb{Z}_n^d$ with scalar product

$$\langle f, g \rangle_n := n^{-d} \sum_{a \in \mathbb{Z}_n^d} f(a)g(a), \quad f, g \in L^2(\mathbb{Z}_n^d).$$

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Proposition

Assume **(T1)**, **(T2)** and **(C1)**. Then, for each $n \in \mathbb{N}$,

- the following operator is well defined (non-symmetric) bilinear form on $\mathcal{F}^n := \{f \in L^2(\mathbb{Z}_n^d) : \mathcal{E}^n(f, f) < \infty\}$,

$$H^n(f, g) = \frac{1}{2}\mathcal{E}^n(f, g) - n^{-d} \sum_{a\in\mathbb{Z}_n^d} \sum_{b\in\mathbb{Z}_n^d} (f(b) - f(a))g(b)C_a^n(a, b);$$

Dirichlet form approach

Proposition (continued)

- $(H^n, \mathcal{F}^n)$ is a regular lower bounded coercive semi-Dirichlet form;

Dirichlet form approach

Proposition (continued)

- $(H^n, \mathcal{F}^n)$ is a regular lower bounded coercive semi-Dirichlet form;
- for any $f \in C_c(\mathbb{Z}_n^d)$ and $g \in \mathcal{F}^n$, it holds $f \in \mathcal{D}_{\mathcal{A}^n}$, $\mathcal{A}^n f \in L^2(\mathbb{Z}_n^d)$ and

$$H^n(f, g) = \langle -\mathcal{A}^n f, g \rangle_n;$$

Dirichlet form approach

Proposition (continued)

- $(H^n, \mathcal{F}^n)$ is a regular lower bounded coercive semi-Dirichlet form;
- for any $f \in C_c(\mathbb{Z}_n^d)$ and $g \in \mathcal{F}^n$, it holds $f \in \mathcal{D}_{\mathcal{A}^n}$, $\mathcal{A}^n f \in L^2(\mathbb{Z}_n^d)$ and

$$H^n(f,g) = \langle -\mathcal{A}^n f, g \rangle_n;$$

- for all $f \in \mathcal{F}^n$,

$$\frac{1}{4}(1 \wedge \alpha_0^n)\mathcal{E}_1^n(f) \leq H_{\alpha_0}^n(f) \leq \frac{2+\sqrt{2}}{2}(1 \vee \alpha_0^n)\,\mathcal{E}_1^n(f)$$

  and

$$(1 \wedge \alpha_0^n)H_1^n(f) \leq H_{\alpha_0}^n(f) \leq (1 \vee \alpha_0^n)\,H_1^n(f).$$

Dirichlet form approach

Let $r_n : L^2(\mathbb{R}^d, dx) \longrightarrow L^2(\mathbb{Z}_n^d)$ and $e_n : L^2(\mathbb{Z}_n^d) \longrightarrow L^2(\mathbb{R}^d, dx)$, $n \in \mathbb{N}$, denote the restriction and extension operators, respectively, defined as follows

$$r_n f(a) = n^d \int_{\bar{a}} f(x)dx, \quad a \in \mathbb{Z}_n^d$$
$$e_n f(x) = f(a), \quad x \in \bar{a}.$$

Dirichlet form approach

Let $r_n : L^2(\mathbb{R}^d, dx) \longrightarrow L^2(\mathbb{Z}_n^d)$ and $e_n : L^2(\mathbb{Z}_n^d) \longrightarrow L^2(\mathbb{R}^d, dx)$, $n \in \mathbb{N}$, denote the restriction and extension operators, respectively, defined as follows

$$\begin{aligned} r_n f(a) &= n^d \int_{\bar{a}} f(x)dx, \quad a \in \mathbb{Z}_n^d \\ e_n f(x) &= f(a), \quad x \in \bar{a}. \end{aligned}$$

We say that $f_n \in L^2(\mathbb{Z}_n^d)$, $n \in \mathbb{N}$, converge strongly to $f \in L^2(\mathbb{R}^d, dx)$ if

$$\lim_{n \nearrow \infty} \|e_n f_n - f\|_{L^2} = 0,$$

and they converge weakly if

$$\lim_{n \nearrow \infty} \langle e_n f_n, g \rangle = \langle f, g \rangle, \quad g \in L^2(\mathbb{R}^d, dx).$$

Dirichlet form approach

Theorem

Assume **(T1)**-**(T4)**, **(C1)** and that $\{P_t^n r_n f\}_{n\geq 1}$ converges strongly to $P_t f$ for all $t \geq 0$ and all $f \in L^2(\mathbb{R}^d, dx)$. Then, there exists a Lebesgue measure zero set, say B, such that for any initial distribution $\mu(dx)$ of $\{X_t\}_{t\geq 0}$ with $\mu(B) = 0$ and any sequence of initial distributions of $\{X_t^n\}_{t\geq 0}$, $n \in \mathbb{N}$, converging weakly to $\mu(dx)$,

$$\{X_t^n\}_{t\geq 0} \xrightarrow[n\nearrow\infty]{d} \{X_t\}_{t\geq 0}.$$

Dirichlet form approach

Definition

Let $\mathcal{C}$ be dense in $(\mathcal{F}, H_1^{1/2})$. Assume the following

(i) for every sequence $\{f_n\}_{n\geq 1}$, $f_n \in \mathcal{F}^n$, converging weakly to some $f \in L^2(\mathbb{R}^d, dx)$ and satisfying $\liminf_{n\nearrow\infty} H_1^n(f_n) < \infty$, we have that $f \in \mathcal{F}$;

(ii) for any $g \in \mathcal{C}$, any $f \in \mathcal{F}$ and any sequence $\{f_n\}_{n\geq 1}$, $f_n \in \mathcal{F}^n$, converging weakly to f, there exists a sequence $g_n \in \mathcal{F}^n$ converging strongly to g and

$$\lim_{n\nearrow\infty} H^n(g_n, f_n) = H(g, f).$$

Then, we say that the forms H^n, $n \in \mathbb{N}$, converge in generalized (Mosco) sense to H.

Dirichlet form approach

If

(C2)
$$x \longmapsto \int_{B_1(0)} |y|^2 k_s(x, x+y)\, dy \in L^2_{loc}(\mathbb{R}^d, dx),$$

$$x \longmapsto \int_{B_1^c(0)} k_s(x, x+y)\, dy \in L^2(\mathbb{R}^d, dx) \cup L^\infty(\mathbb{R}^d, dx),$$

$$x \longmapsto \int_{B_1(0)} |y|(|k(x, x+y) - k(x, x-y)| + |k(x+y, x) - k(x-y, x)|)\, dy \in L^2_{loc}(\mathbb{R}^d, dx),$$

then (under (C1)) for the generator $(\mathcal{A}, \mathcal{D}_\mathcal{A})$ of $\{P_t\}_{t\geq 0}$ (or, equivalently, of $(H, \mathcal{F})$) it holds that

Dirichlet form approach

- $C_c^\infty(\mathbb{R}^d) \subseteq \mathcal{D}_{\mathcal{A}}$;
- for every $g \in C_c^\infty(\mathbb{R}^d)$,

$$\mathcal{A}g(x) = \int_{\mathbb{R}^d} (g(x+y) - g(x) - \langle \nabla g(x), y\rangle 1_{B_1(0)}(y))k(x, x+y)\, dy + \frac{1}{2}\int_{B_1(0)} \langle \nabla g(x), y\rangle (k(x, x+y) - k(x, x-y))\, dy;$$

- for all $g \in C_c^\infty(\mathbb{R}^d)$ and all $f \in \mathcal{F}$,

$$H(g, f) = \langle -\mathcal{A}g, f\rangle.$$

Dirichlet form approach

Theorem

Assume **(T1)**, **(T2)**, **(C1)**, **(C2)**,

(C3) $0 < \liminf_{n \nearrow \infty} \alpha_0^n \leq \limsup_{n \nearrow \infty} \alpha_0^n < \infty$;

(C4) for every $\rho > 0$,

$$\sup_{x \in B_\rho(0)} \int_{\mathbb{R}^d} (1 \wedge |y|^2) k_s(x, x+y) dy < \infty;$$

(C5) for every $\rho > 0$,

$$\limsup_{n \nearrow \infty} \sup_{a \in B_\rho(0)} \sum_{b \in \mathbb{Z}_n^d} (1 \wedge |b|^2) C_s^n(a, a+b) < \infty;$$

Dirichlet form approach

Theorem (continued)

(C6) for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $n_0 \leq m \leq n$ and all $f \in L^2(\mathbb{Z}^d_m)$,

$$\mathcal{E}^n(r_n e_m f, r_n e_m f)^{1/2} \leq \mathcal{E}^m(f,f)^{1/2} + \varepsilon;$$

(C7) for any sufficiently small $\varepsilon > 0$ and large $m \in \mathbb{N}$,

$$\lim_{n \nearrow \infty} \bar{\mathcal{E}}^n_{m,\varepsilon}(f,f) = \mathcal{E}_{m,\varepsilon}(f,f), \quad f \in C_c^{Lip}(\mathbb{R}^d),$$

where

Dirichlet form approach

Theorem (continued)

(C7)

$$\mathcal{E}_{m,\varepsilon}(f,f) := \frac{1}{2}\int_{\{(x,y)\in B_m(0)\times B_m(0):\,|x-y|>\varepsilon\}}(f(y)-f(x))^2 k_s(x,y)dxdy,$$

$$\bar{\mathcal{E}}^n_{m,\varepsilon}(f,f) := \frac{n^d}{2}\int_{\{(x,y)\in B_m(0)\times B_m(0):\,|x-y|>\varepsilon\}}(f(y)-f(x))^2 \bar{C}^n_s(x,y)dxdy,$$

and

$$\bar{C}^n_s(x,y) := \begin{cases} C^n_s(a,b), & x\in\bar{a} \quad \text{and} \quad y\in\bar{b} \\ 0, & x\notin\bar{a} \quad \text{or} \quad y\notin\bar{b}; \end{cases}$$

Dirichlet form approach

Theorem (continued)

(C8) $$\int_{B_1(0)} |y|^2 |k(x, x+y) - n^d C^n([x]_n, [x]_n + [y]_n)| dy \xrightarrow[n\nearrow\infty]{L^2_{loc}(\mathbb{R}^d, dx)} 0;$$

(C9) $$\int_{B_1^c(0)} |k(x, x+y) - n^d C^n([x]_n, [x]_n + [y]_n)| dy \xrightarrow[n\nearrow\infty]{L^2_{loc}(\mathbb{R}^d, dx)} 0;$$

(C10) for all $R > 0$ large enough,

$$\int_{B_{2R}^c(0)} \left(\int_{B_R(-x)} k(x, x+y) dy \right)^2 dx < \infty;$$

Dirichlet form approach

Theorem (continued)

(C11) for all $R > 0$ large enough,

$$\int_{B^c_{2R}(0)} \left(\int_{B_R(-x)} |k(x, x+y) - n^d C^n([x]_n, [x]_n + [y]_n)| dy \right)^2 dx \xrightarrow{n \nearrow \infty} 0;$$

(C12)
$$\int_{B_1(0)} |y||k(x, x+y) - k(x, x-y) - n^d C^n([x]_n, [x]_n + [y]_n) + n^d C^n([x]_n, [x]_n - [y]_n)| dy \xrightarrow[n \nearrow \infty]{L^2_{loc}(\mathbb{R}^d, dx)} 0.$$

The the forms H^n, $n \in \mathbb{N}$, converge to H in Mosco sense.

Dirichlet form approach

For $0 < p \leq 1$ define

$$C^{n,p}(a,b) := \begin{cases} n^d \int_{\bar{a}} \int_{\bar{b}} k(x,y)dxdy, & |a-b| > \frac{2\sqrt{d}}{n^p} \\ 0, & |a-b| \leq \frac{2\sqrt{d}}{n^p}. \end{cases}$$

Dirichlet form approach

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The conditions in (T2)-(T4) will be satisfied if

- for every $\rho > 0$,
$$\sup_{x\in\mathbb{R}^d} \int_{B^c_\rho(x)} k(x,y)dy < \infty;$$

- $\lim_{r\nearrow\infty} \sup_{x\in\mathbb{R}^d} \int_{B^c_r(x)} k(x,y)dy = 0;$

Dirichlet form approach

- there exists $\rho > 0$ such that

$$\limsup_{\varepsilon \searrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{B_\rho(x) \setminus B_\varepsilon(x)} (y_i - x_i) k(x, y) dy \right| < \infty$$

$$\limsup_{\varepsilon \searrow 0} \sup_{x \in \mathbb{R}^d} \int_{B_{\sqrt{d}\varepsilon p} \setminus B_{\sqrt{d}\varepsilon p(x) - (\sqrt{d}/2)\varepsilon}(x)} |y_i - x_i| k(x, y) dx < \infty$$

$$\limsup_{\varepsilon \searrow 0} \varepsilon \sup_{x \in \mathbb{R}^d} \int_{B_\rho(x) \setminus B_{\varepsilon p}(x)} k(x, y) dy < \infty,$$

Dirichlet form approach

- there exists $\rho > 0$ such that

$$\limsup_{\varepsilon \searrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{B_\rho(x) \setminus B_\varepsilon(x)} (y_i - x_i)(y_j - x_j) k(x,y) dy \right| < \infty$$

$$\limsup_{\varepsilon \searrow 0} \sup_{x \in \mathbb{R}^d} \int_{B_{\sqrt{d}\varepsilon p}(x) \setminus B_{\sqrt{d}\varepsilon p - (\sqrt{d}/2)\varepsilon}(x)} |y_i - x_i||y_j - x_j| k(x,y) dx < \infty$$

$$\limsup_{\varepsilon \searrow 0} \varepsilon \sup_{x \in \mathbb{R}^d} \int_{B_\rho(x) \setminus B_{\varepsilon p}(x)} |y_i - x_i| k(x,y) dy < \infty$$

Dirichlet form approach

Examples

- Symmetric processes.

Examples

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- Non-symmetric Lévy processes: Let $B \subseteq \mathbb{R}^d$ be Borel and let $\nu_1(dy) = n_1(y)dy$ and $\nu_2(dy) = n_2(y)dy$ be Lévy measures. Define
$$\nu(dy) := \begin{cases} \nu_1(dy), & y \in B \\ \nu_2(dy), & y \in B^c. \end{cases}$$

Dirichlet form approach

Examples

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- Non-symmetric Lévy processes: Let $B \subseteq \mathbb{R}^d$ be Borel and let $\nu_1(dy) = n_1(y)dy$ and $\nu_2(dy) = n_2(y)dy$ be Lévy measures. Define

$$\nu(dy) := \begin{cases} \nu_1(dy), & y \in B \\ \nu_2(dy), & y \in B^c. \end{cases}$$

- Stable-like processes: Let $\alpha : \mathbb{R}^d \longrightarrow (0,2)$ be Borel measurable such that $0 < \underline{\alpha} \leq \alpha(x) \leq \overline{\alpha} < 2$ and

$$\int_0^1 \frac{(\beta(u)|\log u|)^2}{u^{1+\overline{\alpha}}} du < \infty,$$

where $\beta(u) := \sup_{|x-y|\leq u} |\alpha(x) - \alpha(y)|$.

Dirichlet form approach

Examples

- Then, the (non-symmetric) kernel

$$k(x,y) := \gamma(x)|y-x|^{-\alpha(x)-d}$$

defines a regular lower bounded semi-Dirichlet form whose corresponding Hunt process is called stable-like process. Here

$$\gamma(x) := \alpha(x)2^{\alpha(x)-1}\frac{\Gamma(\alpha(x)/2+d/2)}{\pi^{d/2}\Gamma(1-\alpha(x)/2)}.$$

Thank you for your attention!