Dirichlet heat kernel estimates for symmetric Lévy processes

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Define a Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $\mathbb{R}^d$ as

$$\mathcal{E}(u,v) := \int_{\mathbb{R}^d} \nabla u(x) \cdot A(x) \nabla v(x) dx + \int_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)) J(x,y) dx dy$$

and

$$\mathcal{F} = \{ f \in L^2(\mathbb{R}^d, dx) : \mathcal{E}(f,f) < \infty \}.$$

Here $A(x) = (a_{ij}(x))_{1 \le i,j \le d}$ is a symmetric measurable $d \times d$ matrix-valued function on $\mathbb{R}^d$ and J is a symmetric measurable function on $\mathbb{R}^d \times \mathbb{R}^d \setminus \{x = y\}$.

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Symmetric Hunt process

Under some mild assumptions on J and A, the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is, in fact, a regular Dirichlet form,

i.e., $C_c(\mathbb{R}^d) \cap \mathcal{F}$ is dense in $\mathcal{F}$ with the norm $\mathcal{E}(f,f) + \int_{\mathbb{R}^d} |f|^2 dx$ and $C_c(\mathbb{R}^d) \cap \mathcal{F}$ is dense in $C_c(\mathbb{R}^d)$ with the uniform norm.

So, by Fukushima (71) and Silverstein (74), there is a symmetric Hunt process X in $\mathbb{R}^d$ associated with $(\mathcal{E}, \mathcal{F})$. Its L^2-infinitesimal generator $\mathcal{L}$ is

$$\mathcal{L}u(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}\right) + \lim_{\varepsilon \downarrow 0} \int_{\{|y-x|>\varepsilon\}} (u(y)-u(x))J(x,y)dy.$$

Under some mild assumptions on J and A, X is conservative and the transition function $\mathbb{P}_x(X_t \in dy)$ is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^d$ so that $\mathbb{P}_x(X_t \in dy) = p(t,x,y)dy$ (Chen & Kumagai (08, 10)).

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Assumption on J in Chen, K & Kumagai (11): (WS) & EXP(β)

It is assumed in Chen, K & Kumagai (11) that

$$\frac{c_1}{|x-y|^d\phi_1(|x-y|)\psi_1(c_2|x-y|)} \le J(x,y) \le \frac{c_3}{|x-y|^d\phi_1(|x-y|)\psi_1(c_4|x-y|)}$$

where

(i) ϕ_1 is increasing on $[0,\infty)$ with $\phi_1(0)=0$, $\phi_1(1)=1$ and

$$c_1\left(\frac{R}{r}\right)^{\alpha_1} \le \frac{\phi_1(R)}{\phi_1(r)} \le c_2\left(\frac{R}{r}\right)^{\alpha_2}, \quad \forall 0<r<R<\infty \qquad \textbf{(WS)}$$

for some $0<\alpha_1\le\alpha_2<2$,

(ii) ψ_1 is increasing on $[0,\infty]$ with $\psi_1(r)=1$ for $0<r\le 1$ and

$$c_1 e^{\gamma_1 r^\beta} \le \psi_1(r) \le c_2 e^{\gamma_2 r^\beta}, \quad \forall 1<r<\infty, \qquad (\mathbf{EXP}(\beta))$$

where $\beta\in[0,\infty]$ and $\gamma_2\ge\gamma_1>0$.

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Assumption on J in Chen, K & Kumagai (11): UJS

We say **UJS** (upper jump smoothness) holds if for a.e. $x, y \in \mathbb{R}^d$,

$$J(x,y) \le \frac{c}{r^d} \int_{B(x,r)} J(z,y)dz \qquad \text{whenever } r \le \frac{1}{2}|x-y|. \qquad (\mathbf{UJS})$$

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Estimates of $p(t,x,y)$ for $\beta=0$: Chen & Kumagai (08, 10)

(a) When $\beta = 0$ and $A(x) = 0$,

$$p(t,x,y) \asymp \frac{1}{(\phi_1^{-1}(t))^d} \wedge \frac{t}{|x-y|^d \phi_1(|x-y|)}$$

for every $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

(b) When $\beta = 0$ and $A(x)$ is uniform elliptic and bounded,

$$p(t,x,y) \asymp$$

$$\left(t^{-d/2} \wedge \phi_1^{-1}(t)^{-d}\right) \wedge \left(t^{-d/2} e^{-c|x-y|^2/t} + \left(\frac{1}{(\phi_1^{-1}(t))^d} \wedge \frac{t}{|x-y|^d \phi_1(|x-y|)}\right)\right)$$

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$\beta > 0$ and $A(x) = 0$: Chen, K & Kumagai (11)

Assume $A(x) = 0$ and **EXP**(β) with $\beta > 0$, **WS** and **UJS** hold.

(a) When $0 < \beta \le 1$

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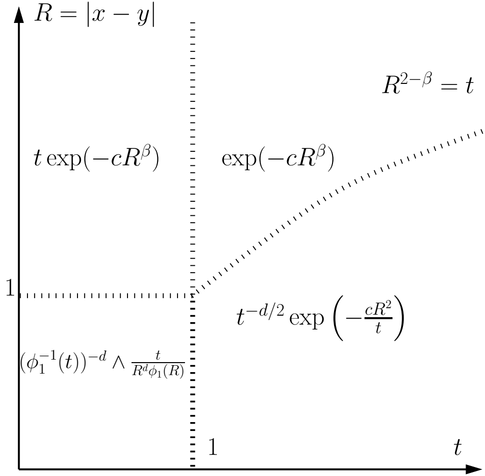
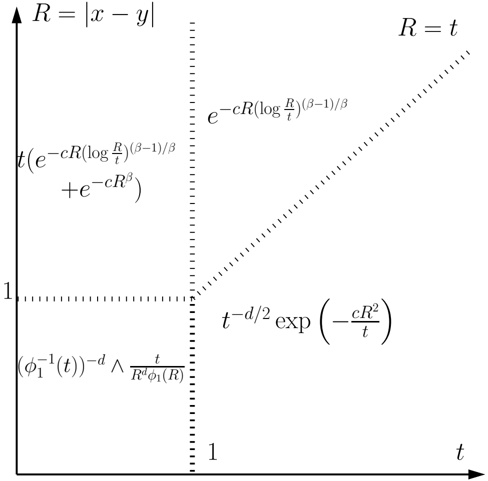
$\beta \in (0,1]$ and $A(x) = 0$



$0 < \beta \leq 1$

$\beta \in (1, \infty)$ and $A(x) = 0$



$1 < \beta < \infty$

$\beta = \infty$ and $A(x) = 0$: Chen, K & Kumagai (11)

For the finite range case i.e. when

$$J(x,y) \asymp |x-y|^{-d}\phi_1(|x-y|)^{-1}1_{\{|x-y|\le 1\}},$$

$$p(t,x,y) \asymp \begin{cases} \dfrac{1}{(\phi_1^{-1}(t))^d} \wedge \dfrac{t}{|x-y|^d\phi_1(|x-y|)}, & \forall t\in(0,1], |x-y|<1; \\ e^{-c|x-y|(1+\log^+\frac{|x-y|}{t})}, & \forall |x-y|\ge t; \\ t^{-d/2}\exp\left(-c\dfrac{|x-y|^2}{t}\right), & \forall t>1, |x-y|\le t. \end{cases}$$

$\beta = \infty$ and $A(x) = 0$: Chen, K & Kumagai (11)

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$\beta > 0$ and $A(x)$ is uniform elliptic: Chen, K & Kumagai (preparation)

Assume $A(x)$ is uniform elliptic and bounded, **EXP**(β) with $\beta > 0$, **WS** and **UJS** hold. Let

$$p^c(t,r) := t^{-d/2}\exp(-r^2/t) \quad \text{and} \quad p^j(t,r) := \left(\phi_1^{-1}(t)^{-d} \wedge \frac{t}{r^d\phi_1(r)\psi_1(r)}\right).$$

(a) When $0 < \beta \le 1$

$$p(t,x,y) \asymp \begin{cases} t^{-d/2} \wedge \left(p^c(t,c|x-y|) + p^j(t,c|x-y|)\right), & \forall t \in (0,1] \\ t^{-d/2}\, e^{-c\left(|x-y|^\beta \wedge \frac{|x-y|^2}{t}\right)}, & \forall t > 1. \end{cases}$$

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Remark

- As you have seen, the explicit estimates of the transition density $p(t, x, y)$ in $\mathbb{R}^d$ depend heavily on the corresponding jumping kernel and the existence of Gaussian component.
- On the other hand, scale-invariant parabolic Harnack inequality holds with the explicit scaling in terms of ϕ_1, $\mathbf{1}_{\{\beta=0\}}$ and the bound in uniform ellipticity condition.
- In particular, if X is a symmetric Lévy processes with above assumptions, the scale-invariant parabolic Harnack inequality holds with the explicit scaling in terms of its Lévy exponent $\Psi(\xi)$ (which you will see in next two slides).

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Lévy processes case

Let $\Psi(\xi)$ be the Lévy exponent of symmetric Lévy process X and use Φ to denote the non-decreasing function

$$\Phi(r) = \frac{1}{\sup_{|z| \le r^{-1}} \Psi(z)}, \qquad r > 0.$$

Lemma

Suppose that X is a symmetric Lévy processes with **EXP**(β) *and* **WS**, *and that a_0 is the bound in uniform ellipticity condition.*

(a) When $\beta = 0$ and $a_0 = 0$, then $\quad \Phi(r) \asymp \phi_1(r)$.

(b) When $\beta = 0$ and $a_0 > 0$, then $\quad \Phi(r) \asymp \begin{cases} r^2 & \text{for } r \in [0,1], \\ \phi_1(r) & \text{for } r \ge 1. \end{cases}$

(c) When $\beta \in (0, \infty]$ and $a_0 = 0$, then $\quad \Phi(r) \asymp \begin{cases} \phi_1(r) & \text{for } r \in [0,1], \\ r^2 & \text{for } r \ge 1. \end{cases}$

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Let $\Psi(\xi)$ be the Lévy exponent of symmetric Lévy process X and use Φ to denote the non-decreasing function

$$\Phi(r) = \frac{1}{\sup_{|z| \le r^{-1}} \Psi(z)}, \qquad r > 0.$$

Lemma

Suppose that X is a symmetric Lévy processes with **EXP**(β) *and* **WS**, *and that a_0 is the bound in uniform ellipticity condition.*

(a) When $\beta = 0$ and $a_0 = 0$, then $\Phi(r) \asymp \phi_1(r)$.

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If X is a symmetric Lévy processes with **EXP**(β), **WS** and **UJS**, then Parabolic Harnack inequality holds with the explicit scaling in terms of Lévy exponent $\Psi(\xi)$;

For every $\delta \in (0,1)$, there exists $c = c(d,\delta) > 0$ such that for every $t_0 \geq 0$, $R > 0$ and every non-negative function u on $[0,\infty) \times \mathbb{R}^d$ that is parabolic on $(t_0, t_0 + 4\delta\Phi(R)] \times B(0,R)$,

$$\sup_{(t_1,y_1)\in Q_-} u(t_1,y_1) \leq c \inf_{(t_2,y_2)\in Q_+} u(t_2,y_2),$$

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Dirichlet heat kernel for Brownian motion

Due to the complication near the boundary, two-sided estimates on the transition density $p_D(t,x,y)$ of killed diffusions in a domain D (equivalently, the Dirichlet heat kernel) have been established recently.

Davies (87), Zhang (02)

Let D be a bounded $C^{1,1}$ domain of $\mathbb{R}^d$ and $\delta_D(x)$ the distance between x and D^c. Let $p_D(t,x,y)$ be the transition density of killed Brownian motion in a domain D

For every $T>0$, there exist $c_1,c_2,c_3,c_4>0$ such that on $(0,T]\times D\times D$

$$c_1(1\wedge\frac{\delta_D(x)}{\sqrt{t}})(1\wedge\frac{\delta_D(y)}{\sqrt{t}})t^{-d/2}e^{-c_2|x-y|^2/t}$$

$$\le p_D(t,x,y)\le c_3(1\wedge\frac{\delta_D(x)}{\sqrt{t}})(1\wedge\frac{\delta_D(y)}{\sqrt{t}})t^{-d/2}e^{-c_4|x-y|^2/t}.$$

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Dirichlet Heat Kernel Estimates of Discontinous Markov Process

Let Y be a Markov process in $\mathbb{R}^d$ whose infinitesimal generator is $\mathcal{L}$ and we assume $\mathbb{P}_x(Y_t \in dy)$ is absolutely continuous with respect to Lebesgue measure in $\mathbb{R}^d$.

For any open subset $D \subset \mathbb{R}^d$, let Y^D be a subprocess of Y killed upon leaving D and $p_D(t,x,y)$ be a transition density of Y^D. $p_D(t,x,y)$ describes the distribution of Y^D: i.e., $\mathbb{P}_x(Y_t^D \in A) = \int_A p_D(t,x,y)dy$.

An infinitesimal generator $\mathcal{L}|_D$ of Y^D is the infinitesimal generator $\mathcal{L}$ with zero exterior condition.

$p_D(t,x,y)$ is also called the Dirichlet heat kernel for $\mathcal{L}|_D$ since

$$u(t,x) := \mathbb{E}_x[f(X_t)] = \int_{\mathbb{R}^d} p_D(t,x,y)f(y)dy$$

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What is the sharp estimates on $p_D(t,x,y)$?

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Theorem. Let D be a $C^{1,1}$ open subset of $\mathbb{R}^d$.

Suppose that $p_D(t,x,y)$ is Dirichlet heat kernel of the fractional Laplacian $-(-\Delta)^{\alpha/2}|_D$ in D.

(i) For every $T>0$, on $(0,T]\times D\times D$

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Since then Dirichlet heat kernel estimates (DHKE) has been generalized to more general stochastic processes:

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Chen & Tokle(11), Chen, K & Song(12), Chen, K & Song(12) : DHKE for stable ($-(-\Delta)^{\alpha/2}$) and relativistic stable ($m-(m^{2/\alpha}-\Delta)^{\alpha/2}$) processes are obtained for all $t>0$ in two classes of unbounded open sets: half-space-like $C^{1,1}$ open sets and exterior $C^{1,1}$ open sets.

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We conjectured in Chen, K and Song(12) that, when D is half space-like $C^{1,1}$ open set, the following two sided estimates holds for a large class of rotationally symmetric Lévy process whose Lévy exponent is $\Psi(|\xi|)$:

there are constants $c_1, c_2, c_3 \ge 1$ such that for every $(t, x, y) \in (0, \infty) \times D \times D$,

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$$\begin{aligned} \frac{1}{c_1} \left(\frac{1}{\sqrt{t\,\Psi(1/\delta_D(x))}} \wedge 1 \right) \left(\frac{1}{\sqrt{t\,\Psi(1/\delta_D(y))}} \wedge 1 \right) p(t, c_2(y-x)) \;\le\; p_D(t,x,y) \\ \le\; c_1 \left(\frac{1}{\sqrt{t\,\Psi(1/\delta_D(x))}} \wedge 1 \right) \left(\frac{1}{\sqrt{t\,\Psi(1/\delta_D(y))}} \wedge 1 \right) p(t, c_3(y-x)) \end{aligned} \tag{1}$$

where $p(t, x)$ is the transition density of X.

This conjecture was proved for rotationally symmetric unimodal pure jump Lévy process with **WS** in Bogdan, Grzywny & Ryznar (14).

An open set D is said to be half-space-like if, after isometry, there exist two real numbers $b_1 \le b_2$ such that

$$\{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > b_1\} \subset D \subset \{x = (\widetilde{x}, x_d) \in \mathbb{R}^d : x_d > b_2\}.$$

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This conjecture was proved for rotationally symmetric unimodal pure jump Lévy process with **WS** in Bogdan, Grzywny & Ryznar (14).

Let us focus on the conjecture (1) when $D = \mathbb{H}$ is a half space. In Chen & K (16), we show that (1) holds for a large class of (not necessarily rotationally) symmetric Lévy processes. Our symmetric Lévy processes may or may not have Gaussian component.

Once the global heat kernel estimates are obtained on upper half space $\mathbb{H}$, one can then use the "push inward" method introduced in Chen & Tokle(11) to extend the results to half-space-like $C^{1,1}$ open sets. See Bogdan, Grzywny & Ryznar (14).

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$d \geq 1$ and $X = (X_t, \mathbb{P}_x)_{t \geq 0, x \in \mathbb{R}^d}$ is a symmetric discontinuous Lévy process (but possibly with Gaussian component) on $\mathbb{R}^d$ with unbounded Lévy exponent $\Psi(\xi)$ and Léve density J where $\mathbb{P}_x(X_0 = x) = 1$.

That is, X is a right continuous symmetric process having independent stationary increments with

$$\mathbb{E}_x\left[e^{i\xi\cdot(X_t - X_0)}\right] = e^{-t\Psi(\xi)} \qquad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d$$

and

$$\Psi(\xi) = \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j + \int_{\mathbb{R}^d}(1 - \cos(\xi\cdot y))J(y)dy \qquad \text{for } \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d,$$

where $A = (a_{ij})$ is a constant, symmetric, non-negative definite matrix and J is a symmetric non-negative function on $\mathbb{R}^d \setminus \{0\}$ with $\int_{\mathbb{R}^d}(1 \wedge |z|^2)J(z)dz < \infty$.

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Assumption (Comp)

Note that X_t^d is a 1-dimensional symmetric Lévy process with Lévy exponent $\Psi_1(\eta) := \Psi((0,\dots,0,\eta))$.

From now on we assume that there exists $c \geq 1$ so that

$$\sup_{|z|\leq r} \Psi(z) \leq c \sup_{|\eta|\leq r} \Psi((0,\dots,0,\eta)) \quad \text{for all } r>0. \qquad (\mathbf{Comp})$$

Condition (**Comp**) holds if X is weakly comparable to an isotropic Lévy process in the following sense.

Suppose there are a non-negative function j on $(0,\infty)$ and $a \geq 0$ $c_i > 0$, $i = 1,2,3$ such that, for all $y \in \mathbb{R}^d$

$$c_1^{-1} a|y|^2 \leq \sum_{i,j=1}^d a_{i,j} y_i y_j \leq c_1 a|\xi|^2 \quad \text{and} \quad c_1^{-1} j(c_2|y|) \leq J(y) \leq c_1 j(c_3|y|),$$

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Recall that

$$\Phi(r) = \frac{1}{\sup_{|z|\le r^{-1}} \Psi(z)}, \qquad r>0.$$

The right continuos inverse function of Φ is $\Phi^{-1}(r)$.

Lemma (Kwaśnicki, Małecki & Ryznar (13) and Bogdan, Grzywny & Ryznar (14))

There exists $C_0>0$ such that

$$C_0^{-1}\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right) \le \mathbb{P}_x(\tau_{\mathbb{H}}>t) \le C_0\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right).$$

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Assumption (ExpL)

We now assume that

$$\int_{\mathbb{R}^d} \exp\left(-t\Psi(\xi)\right) d\xi < \infty \qquad \text{for } t > 0. \qquad \textbf{(ExpL)}$$

Under this condition, the transition density $p(t,x,y) = p(t, y-x)$ of X exists as a bounded continuous function for each fixed $t > 0$, and it is given by

$$p(t,x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t\Psi(\xi)} d\xi, \quad t > 0.$$

Clearly

$$p(t,x) \le (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-t\Psi(\xi)} d\xi = p(t,0) < \infty.$$

Note that (**ExpL**) always holds if $\|A\| > 0$.

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For an open set D, define

$$p_D(t,x,y) := p(t,x-y) - \mathbb{E}_x[\, p(t-\tau_D, Y_{\tau_D} - y) : \tau_D < t] \quad \text{for } t>0, x,y \in D$$

Using the strong Markov property of X, it is easy to verify that $p_D(t,x,y)$ is the transition density for X^D, the subprocess of X killed upon leaving an open set D.

Lemma

Suppose (**ExpL**) *and* (**Comp**) *hold. For each* $a>0$*, there exists a constant* $c = c(a,\Psi) > 0$ *such that for every* $(t,x,y) \in (0,\infty) \times \mathbb{H} \times \mathbb{H}$ *with* $a\Phi^{-1}(t) \le |x-y|$,

$$p_{\mathbb{H}}(t,x,y) \le c\Big(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\Big) \sup_{(s,z):s\le t, \frac{|x-y|}{2} \le |z-y| \le \frac{3|x-y|}{2}} p_{\mathbb{H}}(s,z,y)$$
$$+ \Big(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\Big)\Big(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1\Big) t \sup_{w:|w|\ge \frac{|x-y|}{3}} J(w).$$

PHI(Φ)

Let $Z_s := (V_s, X_s)$ be a time-space process where $V_s = V_0 - s$.

The law of the time-space process $s \mapsto Z_s$ starting from (t,x) will be denoted as $\mathbb{P}^{(t,x)}$.

We say that a non-negative Borel measurable function $h(t,x)$ on $[0,\infty) \times \mathbb{R}^d$ is *parabolic* (or *caloric*) on $(a,b] \times B(x_0,r)$ if for every relatively compact open subset U of $(a,b] \times B(x_0,r)$, $h(t,x) = \mathbb{E}_{(t,x)}[h(Z_{\tau_U^Z})]$ for every $(t,x) \in U \cap ([0,\infty) \times \mathbb{R}^d)$, where $\tau_U^Z := \inf\{s > 0 : \ Z_s \notin U\}$.

We assume the following (scale-invariant) parabolic Harnack inequality **PHI(Φ)** holds for X; For every $\delta \in (0,1)$, there exists $c = c(d,\delta) > 0$ such that for every $x_0 \in \mathbb{R}^d$, $t_0 \ge 0$, $R > 0$ and every non-negative function u on $[0,\infty) \times \mathbb{R}^d$ that is parabolic on $(t_0, t_0 + 4\delta\Phi(R)] \times B(x_0,R)$,

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where $Q_- = (t_0 + \delta\Phi(R), t_0 + 2\delta\Phi(R)] \times B(x_0, R/2)$ and $Q_+ = [t_0 + 3\delta\Phi(R), t_0 + 4\delta\Phi(R)] \times B(x_0, R/2)$.

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Consequences of PHI(Φ)

First note that

$$p(t,0) \le c_1 \inf_{\Phi^{-1}(t)\ge|z|} p(3t,0,z) \le c_2(\Phi^{-1}(t))^{-d} \int_{B(0,\Phi^{-1}(t))} p(3t,0,z)dz \le c_3(\Phi^{-1}(t))^{-d}.$$

$\mathbf{PHI}(\Phi)$ implies (**ExpL**).

Proposition (Interior near diagonal lower bound)

Suppose $\mathbf{PHI}(\Phi)$ *holds. Let* $a>0$ *be a constant. There exists* $c=c(a)>0$ *such that*

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We assume (**UJS**) holds. i.e., there exists a positive constant c such that for every $y \in \mathbb{R}^d$,

$$J(y) \le \frac{c}{r^d} \int_{B(0,r)} J(y-z)dz \qquad \text{whenever } r \le \frac{1}{2}|y|. \qquad (\mathbf{UJS})$$

Note that (**UJS**) is very mild assumption in our setting. In fact, **UJS** always holds if $J(x) \asymp j(|x|)$ for some non-increasing function j.

Moreover, if J is continuous on $\mathbb{R}^d \setminus \{0\}$, then $\mathbf{PHI}(\Phi)$ implies (**UJS**).

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*Suppose $\mathbf{PHI}(\Phi)$ and (**UJS**) hold. For every $a > 0$, there exists a constant $c = c(a) > 0$ such that*

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$$J(y) \le \frac{c}{r^d} \int_{B(0,r)} J(y-z)dz \qquad \text{whenever } r \le \frac{1}{2}|y|. \qquad (\textbf{UJS})$$

Note that (**UJS**) is very mild assumption in our setting. In fact, **UJS** always holds if $J(x) \asymp j(|x|)$ for some non-increasing function j.

Moreover, if J is continuous on $\mathbb{R}^d \setminus \{0\}$, then $\mathbf{PHI}(\mathbf{\Phi})$ implies (**UJS**).

Proposition (Interior off-diagonal lower bound)

*Suppose $\mathbf{PHI}(\Phi)$ and (**UJS**) hold. For every $a > 0$, there exists a constant $c = c(a) > 0$ such that*

$$p_D(t,x,y) \ge ctJ(x-y)$$

for every $(t,x,y) \in (0,\infty) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \ge a\Phi^{-1}(t)$ and $a\Phi^{-1}(t) \le 4|x-y|$.

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Condition (P)

we assume that $x \to p(t,x)$ is weakly radially decreasing in the following sense.

There exist constants $c>0$ and $C_1, C_2 > 0$ such that

$$p(t,x) \le c p(C_1 t, C_2 y) \qquad \text{for } t \in (0,\infty) \text{ and } |x| \ge |y| > 0. \qquad (\mathbf{P})$$

Theorem

Suppose that conditions **(P)**, **(Comp)**, $\mathbf{PHI}(\mathbf{\Phi})$ *and* **(UJS)** *hold. Then there exist constants* $a_1, M_1, c_1, c_2 > 0$ *such that for every* $(t,x,y) \in (0,\infty) \times \mathbb{H} \times \mathbb{H}$

$$c_1 \Big(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\Big)\Big(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1\Big) p(C_1 t, 6^{-1} C_2 (x-y)) \ge p_{\mathbb{H}}(t,x,y)$$

$$\ge c_2 \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1\right) \times$$

$$\times \begin{cases} \inf\limits_{\substack{(u,v): 2M_1\Phi^{-1}(t) \le |u-v| \le 3|x-y|/2 \\ \Phi(\delta_{\mathbb{H}}(u)) \wedge \Phi(\delta_{\mathbb{H}}(v)) > a_1 t}} p_{\mathbb{H}}(t/3, u, v) & \textit{if } |x-y| > 4M_1\Phi^{-1}(t), \\ (\Phi^{-1}(t))^{-d} & \textit{if } |x-y| \le 4M_1\Phi^{-1}(t). \end{cases}$$

Suppose that ψ_1 is an increasing function on $[0,\infty)$ with $\psi_1(r)=1$ for $0<r\le 1$ and there are constants $a_2 \ge a_2 > 0$, $\gamma_2 \ge \gamma_1 > 0$ and $\beta \in [0,\infty]$ so that

$$a_1 e^{\gamma_1 r^\beta} \le \psi_1(r) \le a_2 e^{\gamma_2 r^\beta} \qquad \text{for every } 1<r<\infty.$$

Suppose that ϕ_1 is a strictly increasing function on $[0,\infty)$ with $\phi_1(0)=0$, $\phi_1(1)=1$ and there exist constants $0<a_3<a_4$ and $0<\beta_1\le\beta_2<2$ so that

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We assume that (**UJS**) holds and that there are constants $\gamma \ge 1$, κ_1, κ_2 and $a_0 \ge 0$ such that

$$\gamma^{-1}a_0|\xi|^2 \le \sum_{i,j=1}^d a_{i,j}\xi_i\xi_j \le \gamma a_0|\xi|^2 \quad \text{for every } \xi\in\mathbb{R}^d, \qquad (\mathbf{E})$$

and

$$\gamma^{-1}\frac{1}{|x|^d\phi_1(|x|)\psi_1(\kappa_2|x|)} \le J(x) \le \gamma\frac{1}{|x|^d\phi_1(|x|)\psi_1(\kappa_1|x|)} \quad \text{for } x\in\mathbb{R}^d. \qquad (\mathbf{J})$$

Note that, (**UJS**) holds if $\kappa_1=\kappa_2$ in (**J**).

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Global sharp heat kernel estimates in $\mathbb{R}^d$

Define

$$p^c(t,r) = t^{-d/2}\exp(-r^2/t).$$

Recall that a_0 is the ellipticity constant in (**E**). For each $a>0$, we define a function $h_a(t,r)$ on $(t,r)\in(0,1]\times[0,\infty)$ as

$$h_a(t,r) := \begin{cases} a_0 p^c(t,ar) + \Phi^{-1}(t)^{-d}\wedge\big(tj(ar)\big) & \text{if } \beta\in[0,1] \text{ or } r\in[0,1],\\ t\exp\left(-a\left(r\left(\log\frac{r}{t}\right)^{(\beta-1)/\beta}\wedge r^\beta\right)\right) & \text{if } \beta\in(1,\infty) \text{ with } r\ge 1,\\ (t/r)^{ar} & \text{if } \beta=\infty \text{ with } r\ge 1; \end{cases}$$

and, for each $a>0$, define a function $k_a(t,r)$ on $(t,r)\in[1,\infty)\times[0,\infty)$ as

$$k_a(t,r) := \begin{cases} \Phi^{-1}(t)^{-d}\wedge[(a_0p^c(t,ar)) + tj(ar)] & \text{if } \beta=0,\\ t^{-d/2}\exp\left(-a(r^\beta\wedge\frac{r^2}{t})\right) & \text{if } \beta\in(0,1],\\ t^{-d/2}\exp\left(-ar\left(\left(1+\log^+\frac{r}{t}\right)^{(\beta-1)/\beta}\wedge\frac{r}{t}\right)\right) & \text{if } \beta\in(1,\infty),\\ t^{-d/2}\exp\left(-ar\left(\left(1+\log^+\frac{r}{t}\right)\wedge\frac{r^2}{t}\right)\right) & \text{if } \beta=\infty. \end{cases}$$

Global sharp heat kernel estimates in $\mathbb{R}^d$

Combining Chen & Kumagai (08, 10) and Chen, K & Kumagai (11, 16+) we have

Theorem

Suppose that (**UJS**), (**E**) *and* (**J**) *hold. Then there are positive constants* c_i, $i = 1, \ldots 6$, *which depend on the ellipticity constant* a_0 *of* (**E**), *such that*

$$c_2^{-1} h_{c_1}(t, |x|) \le p(t,x) \le c_2 h_{c_3}(t, |x|) \quad \text{for every } (t,x) \in (0,1] \times \mathbb{R}^d,$$

and

$$c_4^{-1} k_{c_5}(t, |x|) \le p(t,x) \le c_4\, k_{c_6}(t, |x|) \quad \text{for every } (t,x) \in [1,\infty) \times \mathbb{R}^d.$$

In particular, the condition **(P)** *holds.*

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Global sharp heat kernel estimates in half spaces

Theorem (Chen & K (16), To appear in *Acta Applicandae Mathematicae*)

Suppose that (**UJS**), (**E**) *and* (**J**) *hold. Then there exist* $c_1, c_2, c_3 > 0$ *such that for all* $(t, x, y) \in (0, \infty) \times \mathbb{H} \times \mathbb{H}$,

$$p_{\mathbb{H}}(t,x,y) \leq c_1 \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1\right) \times \begin{cases} h_{c_2}(t, |x-y|/6) & \text{if } t \in (0,1), \\ k_{c_2}(t, |x-y|/6) & \text{if } t \in [1,\infty), \end{cases}$$

and

$$p_{\mathbb{H}}(t,x,y) \geq c_1^{-1} \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1\right) \times \begin{cases} h_{c_3}(t, 3|x-y|/2) & \text{if } t \in (0,1), \\ k_{c_3}(t, 3|x-y|/2) & \text{if } t \in [1,\infty). \end{cases}$$

Therefore,

$$p_{\mathbb{H}}(t,x,y) \asymp p(c_1 t, c_2(x-y)) \left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(x))}{t}} \wedge 1\right)\left(\sqrt{\frac{\Phi(\delta_{\mathbb{H}}(y))}{t}} \wedge 1\right).$$

Subordinate Brownian motion

Let $W = (W_t)$ be a Brownian motion in $\mathbb{R}^d$ and
$S = (S_t)$ an independent subordinator with Laplace exponent φ. i.e.,

$$\mathbb{E}e^{-\lambda S_t} = e^{-t\varphi(\lambda)}.$$

Laplace exponent φ belongs to the class of Bernstein functions, i.e. φ is a C^∞ function such that $(-1)^{n+1}\varphi^{(n)} \le 0$ for all $n \in \mathbb{N}$.

The subordinate Brownian motion $Z = (Z_t)$ is defined by

$$Z_t = W_{S_t}.$$

It is a large class of isotropic Lévy process. For example, if $\varphi(\lambda) = \lambda^{\alpha/2}$, $\alpha \in (0, 2)$ then Z_t is the α-stable process.

The Lévy exponent of subordinate Brownian motion is $\psi(|\xi|) = \varphi(|\xi|^2)$.

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Example

Consider a Bernstein function

$$\varphi(\lambda) = \lambda \log(1 + \frac{1}{\lambda}).$$

(R. Schilling, R. Song and Z. Vondraček, *Bernstein functions*, 2nd ed., 2012)

Then φ varies regularly at 0 with index 1 and we have

$$\varphi(\lambda) - \lambda\varphi'(\lambda) = \lambda/(\lambda+1)$$

so that

$$\lim_{\lambda \downarrow 0} \frac{\varphi(\lambda) - \lambda\varphi'(\lambda)}{\lambda} = 1 \quad \text{and} \quad \frac{\varphi(\lambda)}{\varphi(\lambda) - \lambda\varphi'(\lambda)} \sim \log(1 + \frac{1}{\lambda}), \quad \lambda \downarrow 0.$$

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Heat kernel estimates for Subordinate Brownian motion: Ante Mimica, to appear in *Proceedings of the London Mathematical Society*

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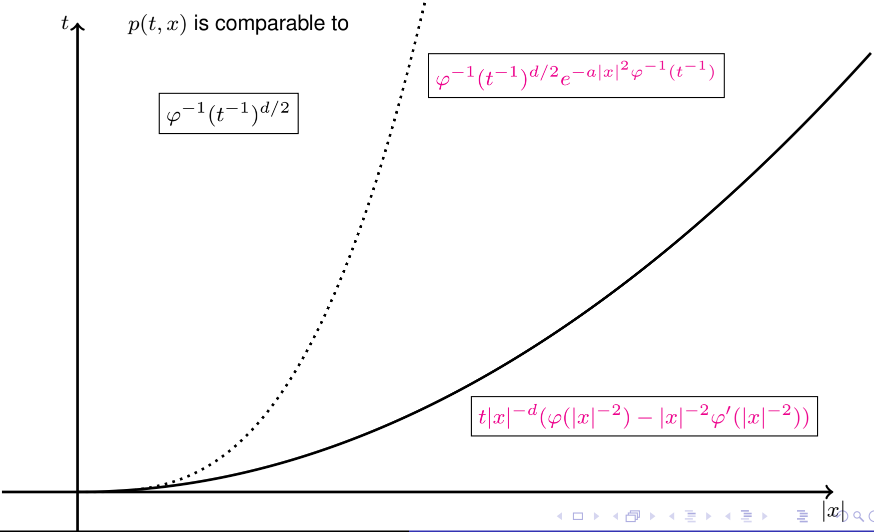

Dirichlet heat kernel estimate for subordinate Brownian motion (Joint work with Ante Mimica, in preparation)

Let

$$H(r) := \phi(r) - r\phi'(r), \qquad r > 0.$$

If there exist $c_1, c_2, \rho > 0$ and $\delta < 2$ such that

$$c_1 \, (R/r)^{\rho} \le H(R)/H(r) \le c_2 \, (R/r)^{\delta}, \quad 0 < r < R < \infty,$$

then for all bounded $C^{1,1}$-domain D and $T > 0$, there exist $C = C(T, D) > 0, a_U, a_L > 0$ such that for any $(t, x, y) \in (0, T] \times D \times D$,

$$p_D(t,x,y) \le C \left(1 \wedge \frac{1}{\sqrt{t\phi(1/\delta_D(x)^2)}}\right) \left(1 \wedge \frac{1}{\sqrt{t\phi(1/\delta_D(y)^2)}}\right) \times \left(\frac{tH(|x-y|^{-2})}{|x-y|^d} + \phi^{-1}(t^{-1})^{d/2} \exp(-a_U |x-y|^2 \phi^{-1}(t^{-1})) \right),$$

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To the memory of Ante Mimica

January 20, 1981 - June 9, 2016