Optimal importance sampling for Lévy processes

Adrien Genin[1,2] P. Tankov[2]

[1]Opus Finance

[2]LPMA, Université Paris Diderot

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Outline

Option pricing with Lévy processes

The financial market consists of n risky assets $S^1, \ldots, S^n$ such that

$$S_t^i = S_0^i e^{X_t^i},$$

where $(X^1, \ldots, X^n)$ is a Lévy process under the risk-neutral probability $\mathbb{P}$.

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- If S is one-dimensional and P depends on S_T only, $\mathbb{E}[P(S)]$ is computed by Fourier transform using the Lévy-Khintchine formula (Carr & Madan '98)
- If the dimension of S is low and path dependence is weak: partial integro-differential equations (Cont & Voltchkova '05), Fourier time stepping (Fang & Oosterlee '08) and related deterministic methods
- High dimension or strong path dependence: Monte Carlo method

Monte Carlo method for Lévy processes

The Monte Carlo method relies on the Law of Large Numbers to simulate the expectation :

$$\widehat{P}_N := \frac{1}{N}\sum_{j=1}^{N} P(S^{(j)}) \to \mathbb{E}\left[P(S)\right], \text{ as } N \to \infty$$

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The precision of standard Monte Carlo is often too low for real-time applications, and various error reduction techniques must be applied

- Multilevel Monte Carlo (Giles '08, Giles & Xia '14 for Lévy models)
- Quasi Monte Carlo (Leobachter '06, Avramidis & L'Ecuyer '06)
- Variance reduction via importance sampling (Badouraly Kassim et al. '15, Guasoni & Robertson '08, Robertson '10, Glasserman et al. '99)

Importance sampling

For any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$,

$$\mathbb{E}[P(S)] = \mathbb{E}^{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}P(S)\right]$$

This leads to the importance sampling estimator

$$\widehat{P}_N^{\mathbb{Q}} := \frac{1}{N}\sum_{j=1}^{N}\left[\frac{d\mathbb{P}}{d\mathbb{Q}}\right]^{(j)} P(S_{\mathbb{Q}}^{(j)}),$$

where $S_{\mathbb{Q}}^{(j)}$ are sample trajectories of S under the measure $\mathbb{Q}$.

For efficient variance reduction, find a probability measure $\mathbb{Q}$ such that S is easy to simulate under $\mathbb{Q}$ and

$$\mathrm{Var}_{\mathbb{Q}}\left[P(S)\frac{d\mathbb{P}}{d\mathbb{Q}}\right] \ll \mathrm{Var}_{\mathbb{P}}\left[P(S)\right].$$

Importance sampling for Lévy processes

For Lévy processes, a natural choice of probability is the Esscher transform

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \frac{e^{\langle \theta, X_T \rangle}}{\mathbb{E}\left[e^{\langle \theta, X_T \rangle}\right]}$$

For path-dependent payoffs, we take the time-dependent Essher transform

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}} = \frac{e^{\int_{[0,T]} X_t \cdot \theta(dt)}}{\mathbb{E}\left[e^{\int_{[0,T]} X_t \cdot \theta(dt)}\right]}$$

where θ is a (deterministic) bounded $\mathbb{R}^n$-valued measure on $[0, T]$. The class of such measures is denoted by M.

Under $\mathbb{P}^\theta$, the process X has independent increments and is thus easy to simulate.

Finding the optimal parameter θ

The optimal choice of θ should minimize the variance of the estimator under $\mathbb{P}^\theta$,

$$\mathrm{Var}_{\mathbb{P}^\theta}\left(P\frac{d\mathbb{P}}{d\mathbb{P}^\theta}\right) = \mathbb{E}_{\mathbb{P}}\left[P^2\frac{d\mathbb{P}}{d\mathbb{P}^\theta}\right] - \mathbb{E}\left[P\right]^2$$

Denoting $H(X) = \log P(S)$, the minimization problem writes

$$\inf_{\theta\in M}\mathbb{E}_{\mathbb{P}}\left[\exp\left\{2H(X) - \int_{[0,T]} X_t\cdot\theta(dt) + \int_0^T G(\theta([t,T]))dt\right\}\right],$$

where

$$G(\theta) = \langle\theta,\mu\rangle + \int_{\mathbb{R}^n}(e^{\langle\theta,x\rangle} - 1 - \langle\theta,x\rangle 1_{|x|\le 1})\nu(dx).$$

Inspired by the works of Glasserman et al. '99 (Gaussian vectors), Guasoni and Robertson '08 (Black-Scholes model), Robertson '10 (stochastic volatility), we approximate the optimal parameter θ^* by minimizing a proxy for the variance computed using the theory of large deviations.

Outline

Large Deviations Principle

Let $\mathcal{X}$ be a Haussdorf topological space endowed with its Borel σ-field.

Definition : Large Deviation Principle

A rate function is a $[0,\infty]$-valued lower semi-continuous function on $\mathcal{X}$. It is said to be a good rate function is its level sets are compact.

A family $\{X^\varepsilon\}$ of $\mathcal{X}$-valued random variables is said to obey a LDP in $\mathcal{X}$ with rate function I if for each open subset $G \subset \mathcal{X}$ and each closed subset $F \subset \mathcal{X}$

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}[X^\varepsilon \in F] \leq -\inf_{x\in F} I(x)$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}[X^\varepsilon \in G] \geq -\inf_{x\in F} I(x)$$

Example: Schilder's theorem

Theorem

- Let $\mathcal{X}$ be the space of continuous paths on $[0, T]$ vanishing at zero endowed with the uniform topology
- Let W be a standard Brownian motion and denote $X^\varepsilon = \sqrt{\varepsilon} W$

Then, (X^ε) satisfies the LDP with good rate function

$$I(x) = \begin{cases} \frac{1}{2}\int_0^T \dot{x}_t^2 dt, & x \text{ abs. cont. with } \int_0^T \dot{x}_t^2 dt < \infty \\ +\infty, & \text{otherwise.} \end{cases}$$

Varadhan's lemma

Varadhan's lemma (extension by Guasoni & Robertson '08)

Let $\{X^\varepsilon\}$ be a family of $\mathcal{X}$-valued random variables satisfying the LDP with a good rate fuction $I : \mathcal{X} \to [-\infty, \infty[$ and let $\phi : \mathcal{X} \to [-\infty, \infty[$ be such that $\{\phi > -\infty\}$ is open and ϕ is continuous on it. Assume further that for some $\gamma > 1$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\left[e^{\frac{\gamma\phi(X^\varepsilon)}{\varepsilon}}\right] < \infty$$

Then,

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\left[e^{\frac{\phi(X^\varepsilon)}{\varepsilon}}\right] = \sup_{x \in \mathcal{X}} \{\phi(x) - I(x)\}$$

Recall the minimization problem

$$\inf_{\theta \in M} \mathbb{E}_{\mathbb{P}}\left[\exp\left\{2H(X) - \int_{[0,T]} X_t \cdot \theta(dt) + \int_0^T G(\theta([t, T]))dt\right\}\right]$$

Notation and topology

- Let D be the space of cadlag paths $x : [0, T] \to \mathbb{R}^n$ with $x(0) = 0$
- Let V_r be the space of cadlag functions on $[0, T]$ with bounded variation
- Let V_r^{ac} be the subspace of V_r consisting of absolutely continuous functions x such that $x_0 = 0$, equipped with the norm $\|x\| = \int_0^T |\dot{x}_s| ds$
- Recall that M denotes the class of bounded $\mathbb{R}^n$-valued measures on $[0, T]$
- Let $\sigma(D, M)$ be the topology on D defined by

$$\lim_n y_n = y \Leftrightarrow \forall \mu \in M, \lim_n \int_{[0,T]} y_n d\mu = \int_{[0,T]} y d\mu.$$

- $\sigma(D, M)$ is stronger than the topology of pointwise convergence but weaker than the uniform topology

Large deviations principle for Lévy processes

Let $X_t^\varepsilon = \varepsilon X_{t/\varepsilon}$ and assume that there is $\lambda_0 > 0$ with $\int_{|x|>1} e^{\lambda_0 |x|}\nu(dx) < \infty$.

Theorem [Leonard, 1999]

The family $\{X^\varepsilon\}$ satisfies the LDP in D for the $\sigma(D, M)$-topology with the good rate function $\bar{J}(y)$ where

$$\bar{J}(x) = \begin{cases} \sup_{\mu \in M} \left\{ \int_{[0,T]} x_t \mu(dt) - \int_0^T G(\mu([t, T]))dt \right\} & \text{if } x \in V_r \\ +\infty & \text{otherwise,} \end{cases}$$

where we recall that

$$G(\lambda) = \log \mathbb{E}\left[e^{\langle \lambda, X_1 \rangle}\right]$$

De Acosta '94 proves an LDP for the uniform topology under the assumption that all exponential moments are finite.

Alternative form of the rate function

Define the Fenchel conjugate of G:

$$L_a(v) = \sup_{\lambda \in \mathbb{R}^d} \{\langle \lambda, v\rangle - G(\lambda)\}$$

and its recession function

$$L_s(v) = \lim_{u\to\infty} \frac{L_a(uv)}{u}.$$

Then,

$$\bar{J}(x) = \begin{cases} \int_{[0,T]} L_a(\frac{d\dot{x}_a}{dt}(t))dt + \int_{[0,T]} L_s(\frac{d\dot{x}_s}{d\mu}(t))d\mu & \text{if } x \in V_r \\ +\infty & \text{otherwise,} \end{cases}$$

where $\dot{x} = \dot{x}_a + \dot{x}_s$ is the decomposition of the measure $\dot{x} \in M$ in absolutely continuous and singular pars with respect to dt and μ in any nonnegative measure on $[0, T]$, with respect to which $\dot{x}_s$ in absolutely continuous.

Outline

A first result

Proposition

Assume that the log payoff H is continuous for the $\sigma(D, M)$-topology (e.g. pointwise continuous) on the open set $\{H > -\infty\}$ and satisfies $H(x) \leq A + B \sup_{s\in[0,T]} |x_s|$. Then, Varadhan's lemma applies with

$$\phi^\theta(X) = 2H(X) - \int_{[0,T]} X_t \cdot \theta(dt) + \int_{[0,T]} G(\theta([t,T]))dt$$

so that

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_{\mathbb{P}}\left[e^{\frac{\phi^\theta(X^\varepsilon)}{\varepsilon}}\right] = \sup_{x\in D}\left\{\phi^\theta(x) - \bar{J}(x)\right\}$$

$$= \sup_{x\in V_r}\left\{\phi^\theta(x) - \int_{[0,T]} L_a(\frac{d\dot{x}_a}{dt}(t))dt - \int_{[0,T]} L_s(\frac{d\dot{x}_s}{d\mu}(t))d\mu\right\}$$

Asymptotic variance reduction

Definition

We say that the variance reduction parameter θ^* is asymptotically optimal if it minimizes

$$\sup_{x\in V_r}\left\{\phi^\theta(x) - \int_{[0,T]} L_a(\frac{d\dot{x}_a}{dt}(t))dt - \int_{[0,T]} L_s(\frac{d\dot{x}_s}{d\mu}(t))d\mu\right\}$$

with

$$\phi^\theta(X) = 2H(X) - \int_{[0,T]} X_t\theta(dt) + \int_{[0,T]} G(\theta([t,T]))dt$$

over $\theta \in M$.

$\Rightarrow$ optimal variance reduction in the large-time asymptotic regime

A more explicit result

Theorem

Let $H : D \to \mathbb{R}_+$ be concave, and let the following assumptions hold true:

- H is upper semicontinuous on V_r^{ac} and for every $x \in V_r$ there is a sequence $\{x_n\} \subset V_r^{ac}$ converging to x in the $\sigma(D, M)$-topology, with $H(x_n) \to H(x)$.
- G is lower semicontinuous and its effective domain is bounded.

Then,

$$\inf_{\theta \in M} \sup_{v \in V_r} \{\phi^\theta(x) - \bar{J}(x)\} = 2 \inf_{\theta \in M} \{\widehat{H}(\theta) + \int_{[0,T]} G(\theta([t, T]))dt\}$$

where

$$\widehat{H}(\theta) = \sup_{x \in V_r} \{H(x) - \int_{[0,T]} x_t \theta(dt)\}.$$

Moreover, if the infimum in the left-hand side of is attained by θ^* then the same θ^* attains the infimum in the right-hand side.

Proof (sketch)

$$\inf_{\theta\in M}\sup_{x\in V_r}\{\phi^\theta(x)-\bar{J}(x)\} = \inf_{\theta\in M}\sup_{x\in V_r}\inf_{\mu\in M}\{2H(X)-\int_{[0,T]}X_t(\theta(dt)+\mu(dt))$$

$$+\int_{[0,T]}G(\theta([t,T]))dt+\int_{[0,T]}G(\mu([t,T]))dt\}$$

$$=\inf_{\theta\in M}\inf_{\mu\in M}\sup_{x\in V_r}\{2H(X)-\int_{[0,T]}X_t(\theta(dt)+\mu(dt))$$

$$+\int_{[0,T]}G(\theta([t,T]))dt+\int_{[0,T]}G(\mu([t,T]))dt\}$$

$$=\inf_{\theta\in M}\inf_{\mu\in M}\{2\widehat{H}\left(\frac{\theta+\mu}{2}\right)+\int_{[0,T]}G(\theta([t,T]))dt+\int_{[0,T]}G(\mu([t,T]))dt\}$$

$$=2\inf_{\theta\in M}\{\widehat{H}(\theta)+\int_{[0,T]}G(\theta([t,T]))dt\},$$

where the last equality follows by convexity of G.

Concavity of log-payoff

Let $\tilde{P}(X) = P(S)$.

Lemma

H is concave whenever $\tilde{P}$ is concave on $\tilde{P} > 0$ and the set $\{\tilde{P} > 0\}$ is convex.

Proof.

Let $0 < \alpha < 1$. Then,

$$\alpha H(a) + (1-\alpha)H(b) = \alpha \log \tilde{P}(a) + (1-\alpha) \log \tilde{P}(b)$$
$$\leq \log(\alpha \tilde{P}(a) + (1-\alpha)\tilde{P}(b)) \leq \log \tilde{P}(\alpha a + (1-\alpha)b) = H(\alpha a + (1-\alpha)b).$$

□

Outline

General European option

Assume that $H\left((x_t)_{0\leq t\leq T}\right) = h(x_T)$ with $h : \mathbb{R}^n \to \mathbb{R}$ concave and continuous. Then, θ is a Dirac measure at T, of size

$$\theta^* = \arg\min_{\theta\in\mathbb{R}^n}\{\hat{h}(\theta) + TG(\theta)\},$$

where $\hat{h}(\theta) = \sup_{v\in\mathbb{R}^n}\{h(v) - v\theta\}$.

- The function $G(\theta)$ is known explicitly in most models
- Under the measure $\mathbb{P}^\theta$, X is still a Lévy process and often falls into the same parametric class since $\mathbb{E}^{\mathbb{P}^\theta}[e^{uX_1}] = e^{G(\theta+u)-G(\theta)}$.
- To compute the optimal parameter θ^*, solve a convex optimization problem in dimension n.

European basket put option

Let $P(S_1, \ldots, S_n) = (K - S_1 - \cdots - S_n)^+$. Then,

$$h(x_1, \ldots, x_n) = \log(K - e^{x_1} - \cdots - e^{x_n})^+.$$

The function $\tilde{P} = (K - e^{x_1} - \cdots - e^{x_n})^+$ is concave on $\{\tilde{P} > 0\}$ by convexity of the exponential and the set $\{e^{x_1} + \ldots e^{x_n} < K\}$ is convex.

The convex conjugate of h is given by

$$\hat{h}(\theta) = \begin{cases} +\infty & \theta_k \geq 0 \text{ for some } k \\ \log \dfrac{K}{1 - \sum_k \theta_k} - \displaystyle\sum_k \theta_k \log \dfrac{-K\theta_k}{1 - \sum_j \theta_j} & \text{otherwise.} \end{cases}$$

Multivariate variance gamma model

Let $b \in \mathbb{R}^n$, Σ be a positive definite $n \times n$ matrix, and define

$$X_t = \mu t + b\Gamma_t + \Sigma W_{\Gamma_t},$$

where Γ is a gamma process with $\mathbb{E}[\Gamma_t] = t$ and $\operatorname{Var}\Gamma_t = t/\lambda$, and μ is chosen to have $\mathbb{E}[e^{X_t^i}] = 1$ for all t and $i = 1, \dots, n$. Then,

$$G(\theta) = \langle \theta, \mu \rangle - \lambda \log \left(1 - \frac{\langle \theta, b \rangle}{\lambda} - \frac{\langle \Sigma\theta, \theta \rangle}{2\lambda} \right), \theta \in \mathbb{R}^n.$$

with

$$\mu^i = \lambda \log \left(1 - \frac{b^i}{\lambda} - \frac{\Sigma_{ii}}{2\lambda} \right), \quad i = 1, \dots, n.$$

Under the measure $\mathbb{P}^\theta$

$$G^\theta(u) = \langle u, \mu \rangle - \lambda \log \left(1 - \frac{\langle u, b + \Sigma\theta \rangle}{\lambda u^*} - \frac{\langle \Sigma u, u \rangle}{2\lambda u^*} \right), \quad u^* = 1 - \frac{\langle \theta, b \rangle}{\lambda} - \frac{\langle \Sigma\theta, \theta \rangle}{2\lambda}.$$

European put in the variance gamma model

In the first example, we let $n = 1$ and price a European put option with pay-off $P(S) = (K - S)^+$.

The model parameters are $\lambda = 1$, $b = -0.2$ and $\sqrt{\Sigma} = 0.2$, which corresponds to annualized volatility of 28%, skewness of -1.77 and excess kurtosis of 2.25.

Variance reduction ratios as function of time to maturity T, for $K = 1$.

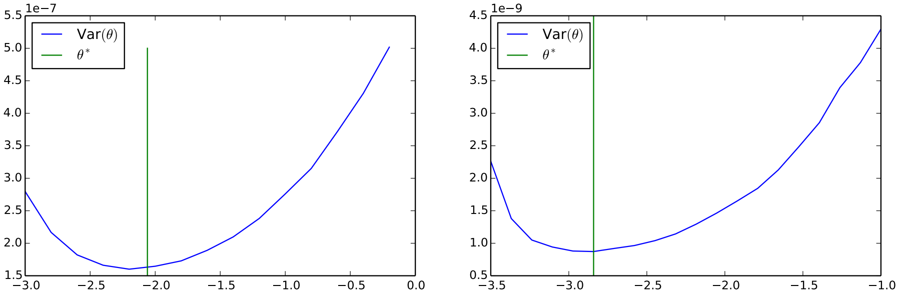
T	0.25	0.5	1	2	3
Optimal parameter θ^*	-2.77	-2.45	-2.06	-1.65	-1.41
Variance ratio	3.38	3.61	3.78	3.75	3.67

Variance reduction ratios as function of strike K, for $T = 1$.

K	0.5	0.7	0.9	1.1	1.3	1.5
Optimal parameter θ^*	-2.84	-2.56	-2.24	-1.88	-1.54	-1.25
Variance ratio	17.44	6.80	4.14	3.19	3.63	3.63

Optimality of θ^*

How optimal is the asymptotically optimal θ^* ?



Left: $T = 1$, $K = 1$. Right: $T = 1$, $K = 0.5$.

Basket put in the variance gamma model

In this example, we let $n = 3$ and price a European basket put option with pay-off $P(S) = (K - S^1 - S^2 - S^3)^+$. The model parameters are

$$\lambda = 1, \quad b = \begin{pmatrix} -0.2 \\ -0.2 \\ -0.2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} 0.04 & 0.02 & 0.02 \\ 0.02 & 0.04 & 0.02 \\ 0.02 & 0.02 & 0.04 \end{pmatrix}.$$

Variance reduction ratios as function of time to maturity T, for $K = 1$.

T	0.25	0.5	1	2	3
Variance ratio	3.55	3.67	3.85	3.81	3.76

K	1.5	2	2.5	3	3.5	4	4.5
Variance ratio, $T = 1$	23.1	9.78	5.53	3.80	3.23	4.22	5.14
Variance ratio, $T = 3$	6.63	4.88	4.35	3.81	2.96	2.42	2.19

Outline

Asian put option: checking the assumptions

For the Asian put option,

$$P(S) = \left(K - \frac{1}{T}\int_0^T S_t dt\right)^+ \quad \Rightarrow \quad H(x) = \log\left(K - \frac{1}{T}\int_0^T e^{x_t} dt\right)^+$$

- $K - \frac{1}{T}\int_0^T e^{x_t} dt$ is concave by convexity of the exponential;
- For x, y such that $\frac{1}{T}\int_0^T e^{x_t} dt < K$ and $\frac{1}{T}\int_0^T e^{y_t} dt < K$,

$$\frac{1}{T}\int_0^T e^{\alpha x_t + (1-\alpha) y_t} dt \leq \frac{1}{T}\int_0^T (\alpha e^{x_t} + (1-\alpha) e^{y_t})\, dt < K,$$

  hence, $\{\tilde{P} > 0\}$ is convex.
- $H(x)$ is usc in the $\sigma(D, M)$ topology but may not be continuous but discretely sampled Asian option is continuous.

Asian put option: computing the Fenchel transform

If θ is absolutely continuous, with density denoted by θ_t, which satisfies $\theta_t \leq 0$ for all $t \in [0, T]$ then the convex conjugate of H is given by

$$\widehat{H}(\theta) = \log \frac{K}{1 - \int_0^T \theta_t dt} - \int_0^T \theta_t \log \frac{-KT\theta_t}{1 - \int_0^T \theta_s ds} dt.$$

Otherwise it is equal to $+\infty$.

Asian put option: optimal importance sampling

To compute the optimal importance sampling measure θ^*, we solve

$$\min_{\theta \in M}\{\widehat{H}(\theta) + \int_0^T G(\theta([t,T]))dt\}$$

$$= \min_{\theta_t \leq 0} \left(1 - \int_0^T \theta_s ds\right) \log \frac{K}{1 - \int_0^T \theta_t dt} - \int_0^T \theta_t \log(-T\theta_t)dt + \int_0^T G\left(\int_t^T \theta_s\right)dt$$

By Pontriagin's principle, $\psi_t = \int_{T-t}^T \theta_s^* ds$ is solution of the system

$$\dot{p}_t = -G'(\psi_t), \qquad p_T = -\log \frac{K}{1-\psi_T} + 1,$$

$$T\dot{\psi}_t = -e^{p_t+1}, \qquad \psi_0 = 0.$$

This can be integrated explicitly:

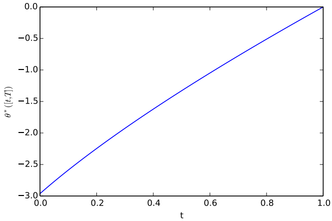
$$\psi_t = F_c^{-1}(-t), \quad \text{with} \quad F(x) = \int_0^x \frac{dy}{c + G(y)}$$

Asian put: numerics

In the same one-dimensional model as above, we price an Asian put option with pay-off $P(S) = \left(K - \frac{1}{T}\int_0^T S_t dt\right)^+$, for $T = 1$.

Variance reduction ratios as function of strike K.

K	0.5	0.7	0.9	1.1	1.3	1.5
Ratio	39.7	10.6	4.82	3.21	5.08	6.91



Optimal θ^* for $K = 1$