Complete Subordinators with nested ranges

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Special Subordinators

Complete subordinators are a subclass of **special** subordinators.

- Analytic definition: a subordinator (S_t) with exponent ψ is special if there exists a subordinator $(\widehat{S}_t)$ with exponent $\widehat{\psi}$ such that for every $\lambda \geq 0$,

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The probabilistic definition only depends on the range $\overline{\{S_t, t \geq 0\}}$ of S, regardless of the time parametrization.

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- (X_t) real-valued Lévy process.
  (S_t) subordinator of ladder times (times at which X reaches a new maximum). Then S is special.
  The dual $\widehat{S}$ is the subordinator of dual ladder times (times at which X reaches a new minimum).

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- Random covering (Fitzsimmons, Fristedt, Shepp).
  $\mathcal{N}$ Poisson point process on $\mathbb{R}_+^2$ with intensity $dt \otimes \mu(dz)$.
  $\mathcal{N}$ generates a random cloud of points (t, z_t).
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$\mathbb{R}^3$ equipped with the lexicographic order:
$(1, 0, 0) \geq (0, 1, 0) \geq (0, 0, 1)$.
(X_t) Lévy process with jumps:

- $(0, 0, 1)$ (rate 1)
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How much can this example be generalized ?

A generalization

- $\mathcal{N}$ Poisson Point process on $\mathbb{R}_+ \times [0, 1]$, intensity $dt \otimes z^{-2}dz$
- $\alpha : [0, 1] \to [0, 1]$ measurable function

From every point (t, h_t) of $\mathcal{N}$ draw a vertical wall up to the x-axis. Colour this wall green w.p. $\alpha(t)$ and red w.p. $1 - \alpha(t)$.

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- If s percolates and $s' < s$, then s' cannot dual-percolate since the highest wall between s' and s is green. Conversely, if s dual-percolates and $s' > s$, s' cannot percolate.

$$G_t \stackrel{distr}{=} t - \widehat{G}_t$$

Nested ranges

Coupling

For every (t, h_t) take an independent, uniform r.v. $U_t \in [0, 1]$. Say that the wall at t is green if $U_t \leq \alpha(t)$ and red otherwise.

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Theorem

The construction above generates the range of a subordinator with exponent

$$\phi^{(\alpha)}(\lambda) = \exp \int_0^1 \frac{(\lambda-1)\alpha(x)}{1+(\lambda-1)x} dx$$

Its dual has exponent $\phi^{(1-\alpha)}$.

One can construct, on a single probability space, a family of random sets $\mathcal{R}^{(\alpha)}$ *indexed by all measurable functions* α*, such that*

- $\mathcal{R}^{(\alpha)}$ *is the range of a subordinator with exponent* $\phi^{(\alpha)}$
- *if* α, β *satisfy, for every* $x \in [0,1]$, $\alpha(x) \leq \beta(x)$, *then* $\mathcal{R}^{(\alpha)} \subset \mathcal{R}^{(\beta)}$.

Some remarks

- One exactly obtains the set of all **complete** subordinators (remark by V. Rivero). A subordinator is complete if its Lévy measure has a completely monotone density.
- When the function α is constant, one gets an α-stable subordinator.
- For constant α, the nested construction corresponds to the Bolthausen-Sznitman coalescent (resp. the Ruelle cascade)
- Multiplying α by a constant $c \in (0,1)$ corresponds to Bochner's subordination by a c-stable subordinator.
- The construction also works in discrete time

Further generalization

The previous construction can be generalized by replacing the horizontal floor by a random floor that has the law of an independent Lévy process. One still obtains a special subordinator whose dual is constructed similarly.
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A simple case

- The floor is deterministic with slope 1.
- $\mathcal{N}$ Poisson Point process on $\mathbb{R}_+ \times [0, 1]$ with intensity $dt \otimes \mu(dz)$.
- All walls are red.

Then s percolates if, from s, one can see no red wall. This is exactly the random covering model.