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Lévy-driven CARMA processes: Non-equidistant observations and local stationarity

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Based on joint work with Annemarie Bitter, Frank Bosserhoff, Żywilla Fechner (Ulm)

Introduction to Lévy-driven CARMA Processes

From ARMA to CARMA

Versatile class of auto-regressive moving-average processes

$$X_n - \varphi_1 X_{n-1} - \ldots - \varphi_p X_{n-p} = \varepsilon_n + \theta_1 \varepsilon_{n-1} + \ldots + \theta_q \varepsilon_{n-q}$$

Extensions to

- multivariate models (Vector ARMA)
- continuous-time models (CARMA)

Advantages of using time series models defined in continuous time:

- Allow handling of irregularly spaced data and missing observations (thus suitable for high-frequency data).
- Allows consistent estimation and inference at different frequencies

Main idea: Difference equations become differential equations

Problem: Definition, Properties and Estimation

CARMA process - definition

A second-order Lévy-driven continuous-time ARMA(p, q) process is defined (see e.g. Brockwell (2001), Brockwell (2009)) in terms of a *state-space representation* of the formal differential equation

$$a(D)Y(t) = b(D)DL(t), \quad t > 0. \tag{1}$$

where D denotes differentiation with respect to t, $(L(t))_{t\geq 0}$ is a Lévy process with $\sigma^2 = \mathbb{E}L(1)^2 < \infty$,

$$a(z) := z^p + a_1 z^{p-1} + \cdots + a_p,$$
$$b(z) := b_0 + b_1 z + \cdots + b_{q-1} z^{q-1} + b_q z^q,$$

with $b_q = 1$ and $q < p$.

The state-space representation

The observation and state equations are given by

$$Y(t) = \mathbf{b}^T \mathbf{X}(t) \tag{2}$$

$$d\mathbf{X}(t) = \mathbf{A}\mathbf{X}(t)dt + \mathbf{e}dL(t) \tag{3}$$

where

$$\mathbf{A} := \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_p & -a_{p-1} & \dots & -a_1 \end{bmatrix}, \quad \mathbf{X}(t) := \begin{bmatrix} X(t) \\ X^{(1)}(t) \\ \vdots \\ X^{(p-2)}(t) \\ X^{(p-1)}(t) \end{bmatrix},$$

$\mathbf{e} := [0, \dots, 0, 1]^T$ and $\mathbf{b} := [b_0, \dots, b_{p-1}]^T$.

Lévy driven CARMA processes

Assumption

$\mathbb{E}[L(1)] = 0$ *and* $\mathbb{E}[L(1)]^2 = \sigma^2 < \infty$.

The solution $\mathbf{X}(t)$ of (3) satisfies

$$\mathbf{X}(t) = e^{\mathbf{A}t}X(0) + \int_0^t e^{\mathbf{A}(t-u)}\mathbf{e}dL(u). \tag{4}$$

If all eigenvalues of $\mathbf{A}$ have strictly negative real parts, the process $(\mathbf{X}(t))_{t\in\mathbb{R}}$ given by

$$\mathbf{X}(t) = \int_{-\infty}^t e^{\mathbf{A}(t-u)}\mathbf{e}dL(u)$$

is a causal strictly stationary solution of (3) for $t \in \mathbb{R}$ with the corresponding CARMA process

$$Y(t) = \int_{-\infty}^t \mathbf{b}^T e^{\mathbf{A}(t-u)}\mathbf{e}dL(u). \tag{5}$$

ACF vs Spectral Density

- The spectral density function

$$f(\omega) := \int_{\mathbb{R}} r(h) e^{-i\omega h} dh$$

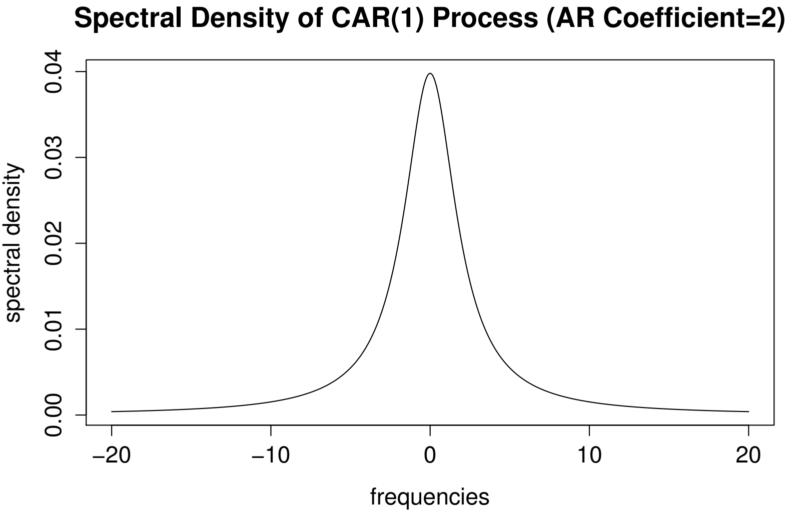
  where $r(h) = \mathbb{C}\mathrm{ov}(Y(0), Y(h))$ denotes the autocovariance function of Y.
- It holds

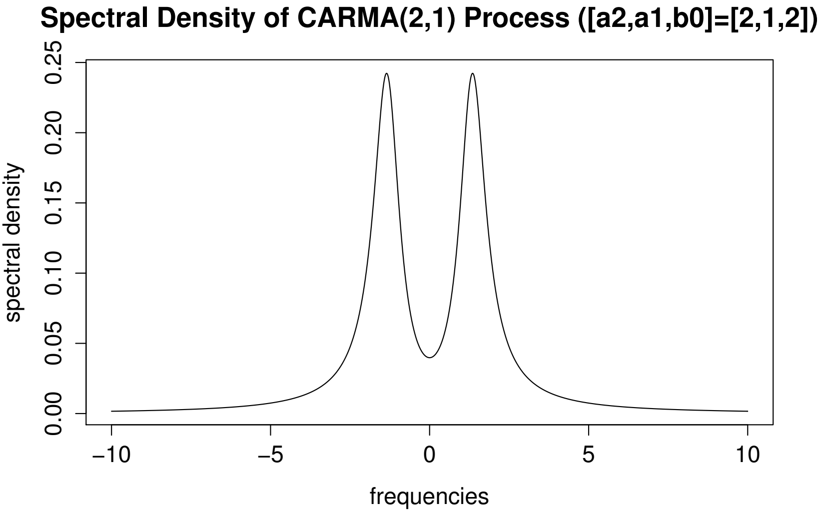
$$r(h) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\omega) e^{i\omega h} d\omega$$

- For CARMA processes the spectral density function is of the form

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{|b(i\omega)|^2}{|a(i\omega)|^2}$$

  and is square integrable.
- The Fourier and inverse Fourier transform: a way of communicating between the time and frequency domain.


Spectral Density of CAR(1) Process (AR Coefficient=2)
spectral density
0.00
0.01
0.02
0.03
0.04
−20
−10
0
10
20
frequencies


Spectral Density of CARMA(2,1) Process ([a2,a1,b0]=[2,1,2])
spectral density
0.00
0.05
0.10
0.15
0.20
0.25
−10
−5
0
5
10
frequencies

Dependence structure

- **Markov properties:**
  As a solution to a stochastic differential equation, the state space representation $\mathbf{X}$ of a causal MCARMA process is a strong Markov process.
- **Mixing properties:**
  For a causal MCARMA process the state space representation $\mathbf{X}$ is β-mixing and Y is strongly mixing, both with exponentially decaying mixing coefficients.
  In particular, $\mathbf{X}$ and Y are ergodic ($\mathbf{X}$ is also geometrically ergodic).

Sample path properties

- The sample paths of a (causal) CARMA(p, q) process Y with $p > q + 1$ are $(p - q - 2)$-times differentiable and the $p - q - 2$th derivative is absolutely continuous.
- If $p = q + 1$, then $\Delta Y(t) = b_q \Delta L(t)$.
- If the driving Lévy process L is a Brownian motion, then the sample paths of Y are continuous and $(p - q - 1)$-times continuously differentiable, provided $p > q + 1$.

Statistics of Lévy-driven CARMA Processes: A Summary

What is known about estimating CARMA processes?

- Brockwell, Davis, and Yang (2011), Schlemm and St. (2012): QML estimation of AR and MA parameters based on equidistant samples for (multivariate) Lévy-driven CARMA processes (Anti-Aliasing Condition needed)
- Fasen and Kimmig (2016): Order selection by information criteria for QML estimators
- Brockwell and Schlemm (2013): reconstruction of the driving Lévy process using a high-frequency limit (for equidistant samples) and based on this GMM estimation of the parameters of the Lévy process (letting the observation time interval to ∞).
- Ferrazzano and Fuchs (2013): Alternative approach to recovering the driving Lévy process for invertible CARMA processes under a high frequency limit

What is known about estimating CARMA processes?

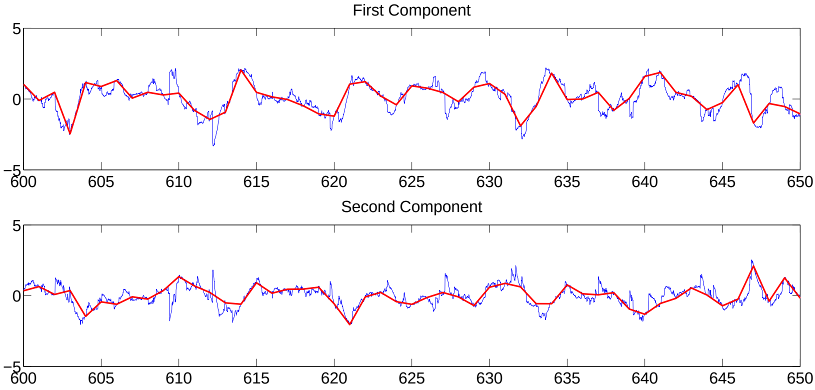
- Fasen and Fuchs (2013b), Fasen and Fuchs (2013a):
  Estimation of power transfer function (also for stable driving Lévy processes) from discrete-time periodograms under a high-frequency infinite time horizon limit using equidistant observations.
- Fasen (2014b), Fasen (2016):
  (high-frequency infinite horizon) limiting results for sample autocovariances
- Gillberg (2006):
  Estimation of non-equidistantly sampled (Brownian) CARMA processes by spline interpolation and using results for equidistant grids

Estimation of Poisson Sampled CARMA Processes

The aliasing problem for equidistant observations

Assume we observe a (multivariate) CARMA process Y at discrete, equally spaced times

$$Y_n^{(h)} := Y(nh), \quad n \in \mathbb{Z}, \quad h > 0.$$



The aliasing problem for equidistant observations

- The autoregressive and moving average parameters and the variance of the Lévy process are identifiable from the spectral density of Y, if the autoregressive and moving average polynomial have no common zeros.
- The autoregressive and moving average parameters and the variance of the Lévy process are NOT identifiable from the spectral density of $Y^{(h)}$, if the autoregressive and moving average polynomial have no common zeros.
- Anti-aliasing condition: All eigenvalues of $\mathbf{A}$ or equivalently all zeros of the autoregressive polynomial are in the set $\{z \in \mathbb{C} : -\pi/h < \Im z < \pi/h\}$.
- Under the anti-aliasing condition, the autoregressive and moving average parameters and the variance of the Lévy process are identifiable from the spectral density of $Y^{(h)}$, if the autoregressive and moving average polynomial have no common zeros.

Poisson Sampling

- Let N be a Poisson process with rate β and jump times $\{\tau_k\}$ which is independent of L.
- The CARMA process is observed at $\{\tau_k\}$ which is equivalent to observing the process Z given by

$$dZ(t) := Y(t)\, dN(t)$$

- Z can also be understood as an orthogonal random measure with spectral density

$$\phi_Z(\omega, \boldsymbol{\theta}) = \beta^2 \left[\frac{\sigma^2}{2\pi} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2 + \frac{\sigma^2 \mathbf{b}^T \Sigma \mathbf{b}}{2\pi\beta} \right] = \frac{\beta^2\sigma^2}{2\pi} \left[\left| \frac{b(i\omega)}{a(i\omega)} \right|^2 + \frac{\mathbf{b}^T \Sigma \mathbf{b}}{\beta} \right].$$

with $\boldsymbol{\theta} = (a_1, \ldots, a_p, b_0, \ldots, b_{q-1})$ and $\Sigma = \int_0^\infty e^{Ay} \mathbf{e}_p\, \mathbf{e}_p^T e^{A^T y}\, dy$.

Identifiability under Poisson Sampling

Proposition (Lii and Masry (1992); Masry (1978))

Assume the autoregressive and moving average polynomials have no common zeros. Then the autoregressive and moving average parameters and the variance of the Lévy process are identifiable from the spectral density of Z.

A Whittle type Estimator

- We assume that we observe Z on $[0, T]$.
- Setting $g_1(\omega, \boldsymbol{\theta}) := \left|\frac{b(i\omega)}{a(i\omega)}\right|^2 + \frac{\mathbf{b}^T\Sigma\mathbf{b}}{\beta}$ and $g_2(\boldsymbol{\theta}) := \frac{\pi \ \mathbf{b}^T\Sigma\mathbf{b}}{\beta} + \int_{-\infty}^{\infty}\left(\frac{|b(ix)/a(ix)|^2}{1+x^2}\right) dx$ we define

$$\hat{K}_T(\boldsymbol{\theta}) = \frac{1}{2\pi T}\int_{-\infty}^{\infty}\left(\frac{\log g_1(\omega,\boldsymbol{\theta}) - \log g_2(\boldsymbol{\theta})}{1+\omega^2}\left|\sum_{k=1}^{N(T)} e^{-i\omega\tau_k} Y(\tau_k)\right|^2\right) d\omega.$$

- Denoting the parameter space of the model by $\boldsymbol{\Theta}$ the Whittle-type estimator is given by:

$$\hat{\boldsymbol{\theta}}_T = \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \hat{K}_T(\boldsymbol{\theta}).$$

- $\boldsymbol{\theta_0}$ denotes the true autoregressive and moving average parameters.

Consistency

Theorem (Bosserhoff, Fechner and St. (2016))

Assume

- $\mathbf{\Theta}$ *is compact,*
- *the driving Lévy process L has finite fourth moments,*
- *for all* $\boldsymbol{\theta} \in \mathbf{\Theta}$ *the autoregressive and moving average polynomials have no common zeros.*

Then

$$\hat{\boldsymbol{\theta}}_T \xrightarrow[T\to\infty]{\mathbb{P}} \boldsymbol{\theta}^{\mathbf{0}}.$$

The Asymptototic Covariance Matrix

- Let $M \in \mathbb{R}^{(p+q)\times(p+q)}$ be defined by

$$M := W^{-1} Q W,$$

where

- $W \in \mathbb{R}^{(p+q)\times(p+q)}$ is given by

$$w_{i,j} = -\int_{-\infty}^{\infty} \frac{\partial \log g(\omega, \boldsymbol{\theta}^{\mathbf{0}})}{\partial \theta_i^0} \frac{\partial \log g(\omega, \boldsymbol{\theta}^{\mathbf{0}})}{\partial \theta_j^0} \frac{\phi_Z(\omega, \boldsymbol{\theta}^{\mathbf{0}})}{1+\omega^2}\, d\omega,$$

- $Q \in \mathbb{R}^{(p+q)\times(p+q)}$ is given by

$$q_{i,j} = 2\pi \Bigg\{ \int_{-\infty}^{\infty} \frac{\partial \log g(\omega, \boldsymbol{\theta}^{\mathbf{0}})}{\partial \theta_i^0} \frac{\partial \log g(\omega, \boldsymbol{\theta}^{\mathbf{0}})}{\partial \theta_j^0} \frac{\phi_Z^2(\omega, \boldsymbol{\theta}^{\mathbf{0}})}{(1+\omega^2)^2}\, d\omega$$
$$+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial \log g(\omega_1, \boldsymbol{\theta}^{\mathbf{0}})}{\partial \theta_i^0} \frac{\partial \log g(\omega_2, \boldsymbol{\theta}^{\mathbf{0}})}{\partial \theta_j^0} \frac{\phi_Z^{(4)}(\omega_1, -\omega_1, \omega_2, \boldsymbol{\theta}^{\mathbf{0}})}{(1+\omega_1^2)(1+\omega_2^2)}\, d\omega_1\, d\omega_2 \Bigg\}.$$

- with $g(\omega, \boldsymbol{\theta}) = g_1(\omega, \boldsymbol{\theta})/g_2(\boldsymbol{\theta})$

Asymptotic Normality

Theorem (Bosserhoff, Fechner and St. (2016))

Assume

- $\mathbf{\Theta}$ *is compact,*
- *the driving Lévy process L has finite moments of all orders,*
- *for all* $\boldsymbol{\theta} \in \mathbf{\Theta}$ *the autoregressive and moving average polynomials have no common zeros,*
- *and the matrix W is invertible.*

Then $\sqrt{T}(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}^{\mathbf{0}})$ *is asymptotically normally distributed with mean zero and covariance matrix M.*

Both consistency and asymptotic normality are proven by verifying the conditions of Lii and Masry (1992) for Lévy-driven CARMA processes.

Estimating the Variance of the driving Lévy Process

Proposition (Bosserhoff, Fechner and St. (2016))

$$\hat{\sigma}_T^2 := \frac{2\pi}{\beta^2} \frac{\int_{-\infty}^{\infty} \frac{I_{Z,T}(\omega)}{1+\omega^2} d\omega}{\frac{\pi \; \hat{\boldsymbol{b}}^T \hat{\Sigma} \hat{\boldsymbol{b}}}{\beta} + \int_{-\infty}^{\infty} \left(\frac{|\hat{b}(i\omega)/\hat{a}(i\omega)|^2}{1+\omega^2} \right) d\omega}$$

is a consistent estimator of σ^2.

Note that we throughout assumed the rate β to be known.

High-frequency irregularly sampled data

Our Approach

- Suppose that we observe a CARMA process at times $t_1 < t_2 < \cdots < t_N$ which are not necessarily uniformly spaced on $[0, T]$. Let
$$h_{\max} := \max_{1 \leq i \leq N-1} (t_{i+1} - t_i)$$
be the maximal distance between observations.
- What can we say about the limiting behaviour of estimators when the length of the interval $T \to \infty$ and the grid size goes to zero (i.e. $h_{\max} \longrightarrow 0$)?

Truncated Fourier Transform

The truncated continuous-time Fourier transform of the process Y at a frequency $\omega \in \mathbb{R}$:

$$\mathcal{F}_T(Y)(\omega) := \frac{1}{\sqrt{T}} \int_0^T Y(t) e^{-i\omega t} dt.$$

Lemma (Fechner and St. (2015))

Let Y be a Lévy-driven CARMA process. Then the truncated Fourier transform of the CARMA *process Y at a fixed frequency $\omega \in \mathbb{R}$ is of the form*

$$\mathcal{F}_T(Y)(\omega) = \frac{1}{\sqrt{T}} \frac{b(i\omega)}{a(i\omega)} \int_0^T e^{-i\omega t} dL(t) + \frac{1}{\sqrt{T}} \mathbf{b}^T (i\omega I - A)^{-1} \left(\mathbf{X}(0) - e^{-i\omega T} \mathbf{X}(T) \right)$$

Truncated Fourier Transform: Covariances

Theorem (Fechner and St. (2015))

Let Y be a Lévy-driven CARMA process. Then for $\omega_1, \omega_2 \in \mathbb{R}$ we have

$$\mathbb{E}\left[\mathcal{F}_T(Y)(\omega_1)\mathcal{F}_T(Y)(\omega_2)\right] = \sigma^2 f_Y(\omega_1) + \frac{1}{T}K(T, \omega_1, \omega_2), \quad \omega_1 = -\omega_2 \tag{6}$$

and

$$\mathbb{E}\left[\mathcal{F}_T(Y)(\omega_1)\mathcal{F}_T(Y)(\omega_2)\right] = \frac{1}{T}K^1(T, \omega_1, \omega_2), \quad \omega_1 \neq -\omega_2, \tag{7}$$

where f_Y is the spectral density function of the process Y, K, K_1 are bounded functions of T.

Truncated Fourier Transform: Asymptotic Normality I

Proposition (Fechner and St. (2015))

Let

$$Z(T) := \frac{1}{\sqrt{T}} \frac{b(0)}{a(0)} \int_0^T dL(t).$$

Then

$$\mathrm{dlim}_{T\to\infty} Z(T) = \mathrm{dlim}_{T\to\infty} \mathcal{F}_T(Y)(0) \sim \mathcal{N}\left(0, \left(\frac{b(0)}{a(0)}\right)^2 \sigma^2\right)$$

$$\mathrm{dlim}_{T\to\infty} \left|\frac{a(0)Z(T)}{b(0)\sigma^2}\right|^2 = \mathrm{dlim}_{T\to\infty} \left|\frac{a(0)\mathcal{F}_T(Y)(0)}{b(0)\sigma^2}\right|^2 \sim \chi^2(1)$$

Truncated Fourier Transform: Asymptotic Normality II

Proposition (Fechner and St. (2015))

Assume that $\omega \neq 0$. *Put*

$$\tilde{Z}(T) := \frac{1}{\sqrt{T}} \frac{b(i\omega)}{a(i\omega)} \int_0^T e^{-i\omega t} dL(t) \text{ and } Z(T) = \begin{bmatrix} \Re \tilde{Z}(T) \\ \Im \tilde{Z}(T) \end{bmatrix}.$$

Then

$$\operatorname{dlim}_{T\to\infty} Z(T) = \operatorname{dlim}_{T\to\infty} \begin{pmatrix} \Re \mathcal{F}_T(Y)(\omega) \\ \Im \mathcal{F}_T(Y)(\omega) \end{pmatrix} \sim \mathcal{N}(0, \Sigma),$$

where

$$\Sigma = \frac{\sigma^2}{2} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The result can be extended to the joint convergence at different frequencies which are asymptotically independent.

Non-equidistant grids: Set-Up

- Assume now that for a time horizon $T > 0$ we observe the CARMA process Y at $N^{(T)}$ non-equidistant time points

$$0 = x_0^{(T)} < x_1^{(T)} < \cdots < x_{N^{(T)}-1}^{(T)} = T.$$

- Let $h_{\max}^{(T)} = \max_{j=0,\dots,N^{(T)}-2}(x_{j+1}^{(T)} - x_j^{(T)})$.
- We denote the trapezoidal approximation of

$$\mathcal{F}_T(Y)(\omega) := \frac{1}{\sqrt{T}} \int_0^T Y(t) e^{-i\omega t} dt$$

by

$$\mathcal{T}_T(Y)(\omega).$$

Non-equidistant grids: Error Bounds

Proposition (Fechner and St. (2015))

1. *We have that*

$$\|\mathcal{T}_T(Y)(\omega) - \mathcal{F}_T(Y)(\omega)\|_{L^2}^2 \leq \frac{C_1 C_2}{T} + C_1 {N^{(T)}}^2 {h_{\max}^{(T)}}^6$$

where the constants C_1, C_2 can be chosen independent of T.

2. *So if $\lim_{T\to\infty} N^{(T)} {h_{\max}^{(T)}}^3 = 0$, then $\mathcal{T}_T(Y)(\omega) - \mathcal{F}_T(Y)(\omega) \to 0$ as $T \to \infty$ in L^2 and thus in probability.*

The proof relies on the smoothness of $e^{-i\omega t}$ and techniques adopted from Brockwell and Schlemm (2013) to non-equidsitant grids and changing time horizons.

Non-equidistant grids: Asymptotic Normality I

Proposition (Fechner and St. (2015))

If $\lim_{T\to\infty} N^{(T)} {h_{\max}^{(T)}}^3 = 0$*, then we have for the trapezoidal approximation of the truncated Fourier transform*

$$\operatorname{dlim}_{T\to\infty} \mathcal{T}_T(Y)(0) \sim \mathcal{N}\left(0, \left(\frac{b(0)}{a(0)}\right)^2 \sigma^2\right)$$

$$\operatorname{dlim}_{T\to\infty} \left|\frac{a(0)\mathcal{T}_T(Y)(0)}{b(0)\sigma^2}\right|^2 \sim \chi^2(1).$$

Non-equidistant grids: Asymptotic Normality II

Proposition (Fechner and St. (2015))

If $\lim_{T\to\infty} N^{(T)} {h_{\max}^{(T)}}^3 = 0$ *and* $\omega \neq 0$, *then we have for the trapezoidal approximation of the truncated Fourier transform*

$$\operatorname{dlim}_{T\to\infty} \begin{pmatrix} \Re \mathcal{T}_T(Y)(\omega) \\ \Im \mathcal{T}_T(Y)(\omega) \end{pmatrix} \sim \mathcal{N}(0, \Sigma),$$

where where

$$\Sigma = \frac{\sigma^2}{2} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The result can again be extended to the joint convergence at different frequencies which are asymptotically independent.

Simulated Example I

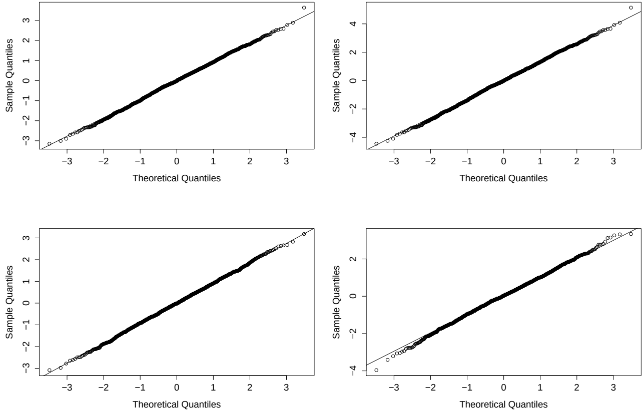


Figure : QQ plots for CAR(1) process driven by standard Brownian Motion noise with coefficients $a_1 = 1$ and $b_0 = 0$. Plotted are 2000 paths observed on a 2000 point (moderately) non-equidistant grid over the interval $[0, 10]$.

Simulated Example II

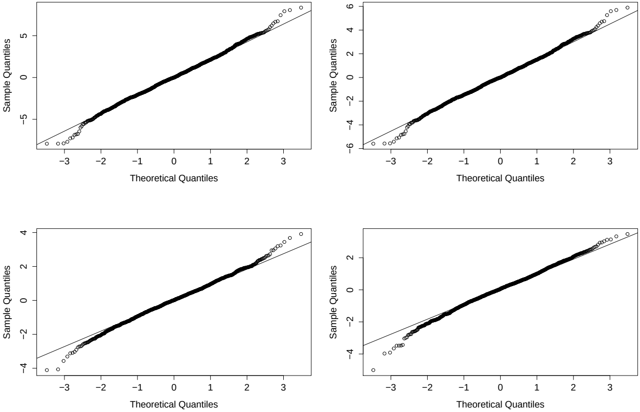


Figure : QQ plots for CARMA(2, 1) process driven by a Variance Gamma noise with coefficients $a_2 = 1$, $a_1 = 2$, $b_1 = 1$. Plotted are 2000 paths observed on a 2000 point (moderately) non-equidistant grid over the interval $[0, 10]$

Locally stationary Lévy-driven CARMA Processes

Motivation

- Often observed time series behave stationary in the short run, but over longer periods the dynamics change.
- Hence *time-varying dynamics are encountered*.
- Various time-varying models have been defined in discrete time, e.g. *tvARMA*, *tvGARCH*.
- Problem: Statistical inference. How to define meaningful estimators and obtain appropriate asymptotics?
- Solution: “in-fill asymptotics” and consideration of sequences of processes
- Intensively studied in discrete time (see e.g. Dahlhaus (2012))

Stationary Moving Average Processes in Continuous Time

- Let L be a second order Lévy process.
- For $g \in L^2$ or $A \in L^2$ a linear/moving average process is given by

$$Y(t) = \int_{\mathbb{R}} g(t-u)dL(u) = \int_{\mathbb{R}} e^{i\mu t} A(\mu)\Phi_L(d\mu).$$

- Φ_L is the orthogonal random measure determined by

$$\Phi_L([a,b)) = \int_{\mathbb{R}} \frac{e^{-ita} - e^{-itb}}{2\pi it} dL(t).$$

- A and g are Fourier transforms of each other:

$$g(t-u) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\mu(t-u)} A(\mu)d\mu \text{ and } A(\mu) = \int_{\mathbb{R}} e^{-i\mu(t-u)} g(t-u)du$$

Local Stationarity in Continuous Time

Definition (Bitter and St. (2016))

A sequence of linear stochastic processes $\{Y_N(t)\}_{N\in\mathbb{N}}$ given by

$$Y_N(t) = \int_{\mathbb{R}} g_N^0(Nt, Nt-u)dL(u) = \int_{\mathbb{R}} e^{i\mu t} A_N^0(Nt, mu)\Phi_L(d\mu)$$

is called locally stationary if one of the following equivalent conditions holds:

- there exists $g : \mathbb{R}^2 \to \mathbb{R}$ continuous in the first coordinate such that $g_N^0(Nt, \cdot) \to g(t, \cdot)$ for $N \to \infty$ in L^2 for all $\in \mathbb{R}$.
- there exists $A : \mathbb{R}^2 \to \mathbb{C}$ continuous in the first coordinate such that $A_N^0(Nt, \cdot) \to A(t, \cdot)$ for $N \to \infty$ in L^2 for all $\in \mathbb{R}$.

Local Stationarity: Immediate Properties

- For each $t \in \mathbb{R}$ $Y_N(t)$ converges in distribution to $\int_{\mathbb{R}} A(t,\mu)\Phi_L(d\mu) = \int_{\mathbb{R}} g(t, t-u)dL(u)$.
- For $t_1 \neq t_2$ $Y_N(t_1)$ and $Y_N(t_2)$ are asymptotically uncorrelated.

Locally Stationary OU type processes

Proposition (Bitter and St. (2016))

The sequence of stochastic processes $\{Y_N(t) : t \in \mathbb{R}\}_{N\in\mathbb{N}}$ *defined by:*

$$Y_N(t) = \int_{-\infty}^{Nt} e^{-\int_u^{Nt} a(\frac{s}{N})ds} L(du), \tag{8}$$

is locally stationary, if the coefficient function $a : \mathbb{R} \mapsto \mathbb{R}$ *satisfies:*

(C1) $a(\cdot)$ *is continuous on* $\mathbb{R}$,

(C2) $a(t) \geq \varepsilon$ *for some* $\varepsilon > 0$.

The limiting kernel / transition matrix is given by:

$$g(t,u) = \mathbf{1}_{\{u\geq 0\}} e^{-a(t)u} \quad \text{and} \quad A(t,\mu) = \frac{1}{a(t) - i\mu}.$$

Note: $dY(t) = a(t)Y(t)dt + L(dt)$ has the solution $Y(t) = \int_{-\infty}^{t} e^{-\int_u^t a(s)ds} L(du)$.

Wigner-Ville Spectrum

Definition

For a locally stationary process the Wigner-Ville spectrum for fixed $N \in \mathbb{N}$ is defined as

$$f_N(t,\lambda) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda s}\mathbb{C}\mathrm{ov}\left(Y_N(t+\tfrac{s}{2N}), Y_N(t-\tfrac{s}{2N})\right) ds$$

and the (time-varying) spectral density of the process $Y_N(t)$ as

$$f(t,\lambda) := \frac{\Sigma_L}{2\pi}|A(t,\lambda)|^2.$$

Convergence of the Wigner-Ville Spectrum

Theorem (Bitter and St. (2016))

Let $\{Y_N(t), t \in \mathbb{R}\}_{N\in\mathbb{N}}$, $Y_N(t) = \int_{\mathbb{R}} e^{i\mu N t} A_N^0(Nt,\mu)\Phi_L(d\mu)$, *be a locally stationary sequence of stochastic processes. Assume*

(i) *For all* $t, s \in \mathbb{R}$ $||A_N^0(N(t \pm \frac{s}{2N}), \cdot) - A(t,\cdot)||_{L^2(\mathbb{R},\mathbb{C})} \xrightarrow{N\to\infty} 0$,

(ii) A_N^0 *and* A *are uniformly bounded in* L^2,

(iii) *For all* t, N, μ *the derivatives of* A_N^0 *and* A *w.r.t.* μ *exist and are uniformly bounded in* L^2.

Then the Wigner-Ville spectrum tends pointwise for each $t \in \mathbb{R}$ *in mean square to the time-varying spectral density:*

$$\int_{\mathbb{R}} |f_N(t,\lambda) - f(t,\lambda)|^2 d\lambda \xrightarrow{N\to\infty} 0.$$

Thank you very much for your attention!

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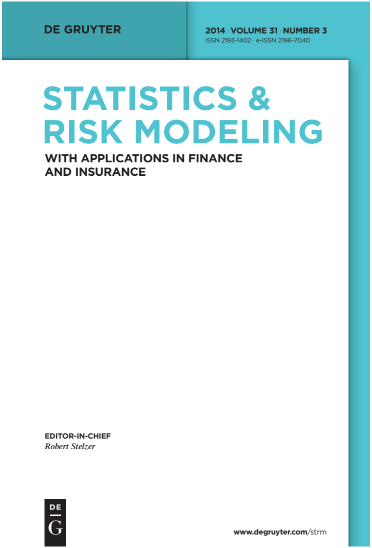

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