The Strong Renewal Theorem via local large deviations

Francesco Caravenna (Milan) and Ron Doney (Manchester)

Introduction

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- The SRT is due to Garsia and Lamperti (1963), for renewal processes, and Williamson (1968) and Erickson (1970) for random walks. It states that when $\alpha \in (1/2, 1)$

$$x\overline{F}(x)g(x) \to c_{\alpha,\rho} = \frac{\int_0^\infty y^{-\alpha}\phi(y)dy}{\Gamma(\alpha)}.$$

- It cannot hold for all cases when $\alpha \leq 1/2$, since

$$x\overline{F}(x)\sum_{1}^{n_0} P(S_n = x) \to 0$$

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- Using Gnedenko's local limit theorem it is straightforward to show that

$$\lim_{\delta\to 0}\lim_{x\to\infty} x\overline{F}(x)\sum_{n>1/\overline{F}(\delta x)} P(S_n = x) = c_{\alpha,\rho},$$

so we are left to prove that $\sum_{n>1/\overline{F}(\delta x)} P(S_n = x)$ is **asymptotically neglible, (a.n.)**, i.e.

$$\lim_{\delta\to 0}\limsup_{x\to\infty} x\overline{F}(x)\sum_{n>1/\overline{F}(\delta x)} P(S_n = x) = 0.$$

Renewal processes

- Our approach is based on large deviations, so with $0 < \lambda < C$ fixed, we put $Z_1 = \max_{r \leq n} X_r$, then consider

$$E_0 = (x - Z_1 \in [\lambda a_n, Ca_n]).$$

We write $A(\cdot)$ for a continous interpolant of $1/\overline{F}(\cdot)$, and a is its inverse, so that a_n is a norming sequence for S_n.

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- Using Gnedenko's local limit theorem again we get

$$\begin{aligned} P(S_n &= x, E_0) \approx n \sum_{\lambda a_n}^{Ca_n} p(x-y) P(S_{n-1} = y) \\ &\approx \frac{n}{a_n} \sum_{\lambda a_n}^{Ca_n} p(x-y), \end{aligned}$$

  so that

-

$$\sum_{1}^{A(\delta x)} P(S_n = x, E_0) \approx \sum_{1}^{A(\delta x)} \frac{n}{a_n} \sum_{\lambda a_n}^{C a_n} p(x-y)$$

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  in all cases.
- We could also consider "2 big jumps", and we would get a bound involving

  $$I_2(\delta, x) := \sum_{1}^{C\delta x} p(x-y_1) \sum_{1}^{\eta y_1} p(y_1 - y_2) \frac{A(y_2)^3}{y_2}.$$

  But in fact

- $$\sum_{1}^{A(\delta x)} P(S_n = x, E_0) \approx \sum_{1}^{A(\delta x)} \frac{n}{a_n} \sum_{\lambda a_n}^{C a_n} p(x-y)$$
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  But in fact
- THEOREM Suppose $\alpha \in (0, \frac{1}{2}]$. Then the SRT holds iff $I(\delta, x)$ is asymptotically neglible.

- The point is that this condition implies that

$$\sum_{1}^{\eta y_1} p(y_1 - y_2)\frac{A(y_2)^3}{y_2} \leq cA(y_1) \cdot \frac{A(y_1)}{y_1},$$

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- In this argument, if we took $\lambda = 0$, we would apparently get a problem when η is integer valued. e.g

$$\text{if } \alpha = 1/2, \sum_{m}^{\infty} n/a_n \not\sim m^2/a_m$$

But this doesn't matter, because for positive summands there is a "small deviations" bound for
$P(S_n = x)$ when x/a_n is small.

Sufficiency

- We need the following local limit results, which also hold in the rw case;
  BOUND 1 Given any $\gamma > 0$, $\exists C_0$, such that, for all n and x,

$$P\{S_n = x, Z_1 \leq \gamma x\} \leq \frac{C_0\{n\overline{F}(x)\}^{1/\gamma}}{a_n}.$$

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- We also have
  $$\begin{aligned} P\{S_n &= x, Z_1 > \gamma x\} \leq n \sum_{z>\gamma x} p(z)P(S_{n-1} = x - z) \\ &\leq \frac{cn\overline{F}(\gamma x)}{a_n}, \end{aligned}$$

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  and taking $\gamma = 1$ we get **BOUND 2**

$$P\{S_n = x\} \leq C_0 \frac{n\overline{F}(x)}{a_n}.$$

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- Here we use the first bound with $\gamma = 1/2$ to get

$$\begin{aligned}
& \limsup_{x\to\infty} \frac{x}{A(x)} \sum_{1}^{A(\delta x)} P_0 \\
\le \; & \limsup_{x\to\infty} \frac{cx\overline{F}(x)^2}{A(x)} \sum_{1}^{A(\delta x)} \frac{n^2}{a_n} \\
\le \; & \limsup_{x\to\infty} \frac{cxA(\delta x)^3}{A(x)^3 a(A(\delta x))} \le \delta^{3\alpha-1}.
\end{aligned}$$

- Next, consider

$$
\begin{aligned}
P_2 \quad : \quad & = P(S_n = x, x/2 < Z_1 \leq x - Ca_n) \\
& \leq n \sum_{Ca_n}^{x/2} p(x-z) P(S_{n-1} = z) \\
& \leq \frac{cn^2}{a_n} \sum_{Ca_n}^{x/2} p(x-z) \overline{F}(z)
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  by bound 2.
- Then a small calculation shows that $\sum_1^{A(\delta x)} P_2$ is also a.n., and this proves the theorem for $1/3 < \alpha \le 1/2$.

- The crucial fact here is that

$$\sum_{1}^{m} n^2/a_n \backsim \frac{cm^3}{a_m}$$

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- This suggests splitting $(S_n = x)$ by considering Z_1 and Z_2 and using the fact that

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- This works....

The random walk case

At first, it looks as if we need only a slight modification of the renewal process results, because by again considering the value of Z_1 we can easily get:

If $\alpha \in (1/3, 1/2)$ the SRT holds iff

$$I_1(\delta, x) := \sum_{|y| \leq \delta x} p(x - y) \frac{A(x)^2}{x} \text{ is a.n.}$$

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- So for example the NASC when $\alpha \in (1/4, 1/3)$ is that $I_2(\delta, x)$ is a.n., where

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- As before, the assumption that $I_1(\delta, x)$ is a.n. implies that the part of this with $y_1 > 0$ is also a.n., but **NOT**

$$\sum_{y_1 \in [-\delta x, 0)} p(x - y_1) \sum_{|y_2| \le \eta |y_1|} p(y_1 - y_2) \frac{A(y_2)^3}{|y_2|}.$$

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- When $\alpha \in (1/5, 1/4)$ the NASC is that $I_3(\delta, x)$ is a.n., where

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- You can guess the form of $I_k(\delta, x)$, the quantity appropriate when $k = k_\alpha := [\eta - 1]$.

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- You can guess the form of $I_k(\delta, x)$, the quantity appropriate when $k = k_\alpha := [\eta - 1]$.
- But there is a further complication for integer values of η. e.g. when $\alpha = 1/2$ the appropriate quantity is

$$\tilde{I}_1(\delta, x) := \sum_{|y| \leq \delta x} p(x - y) \sum_{m = A(|y|)}^{A(\delta x)} \frac{m}{a_m}.$$

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- What about the RW case with $\rho = 0$?
- Do the results extend to Lévy processes?