MAPs, Kuznetsov Measures and Self-Similarity

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joint work with Leif Döring and Andreas Kyprianou

Lévy Processes @ Angers

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0 Positive self-similar Markov processes (pssMp)

Positive self-similar Markov process of index $\alpha > 0$:

- strong càdlàg Markov family $\{\mathbb{P}^z, z \in (0,\infty)\}$ on the state space $[0,\infty)$ - 0 being an absorbing cemetery state - such that the scaling property holds:

$$\text{the law of } (cZ_{c^{-\alpha}t})_{t\geq 0} \text{ under } \mathbb{P}^z \text{ is } \mathbb{P}^{cz} \qquad \text{(SP)}$$

for all $z, c > 0$, where $Z = (Z_t)_{t\geq 0}$ denotes the canonical process.

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for all $z, c > 0$, where $Z = (Z_t)_{t\geq 0}$ denotes the canonical process.

Lamperti representation: (Lamperti '72)

- $\exists$ a Lévy process $\xi = (\xi_t)_{t\geq 0}$ (possibly killed with cemetery state $-\infty$), such that, under $\mathbb{P}^z$ for $z > 0$,

$$Z_t = \exp(\xi_{A(t)}), \qquad t \geq 0,$$

where ξ is started in $\log(z)$ and

$$A(\cdot) = \left(\int_0^{\cdot} \exp(\alpha\xi_s)ds\right)^{-1}.$$

Note: To avoid technicalities we assume that the Lévy process ξ is nonlattice!

0 Two regimes

Two regimes: Let T_0 be first hitting time of zero of Z.

(R) $\mathbb{P}^z(T_0 < \infty) = 1$ for all $z > 0 \;\Leftrightarrow\; \xi$ drifts to $-\infty$ or is killed

(T) $\mathbb{P}^z(T_0 < \infty) = 0$ for all $z > 0 \;\Leftrightarrow\; \xi$ drifts to ∞ or oscillates

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Example: Squared-Bessel processes of dimension $\delta \in \mathbb{R}$

$$dZ_t = 2\sqrt{Z_t}dB_t + \delta dt, \quad t \leq T_0,$$

self-similar of index 1 with corresponding Lévy process

$$\xi_t = 2W_t + (\delta - 2)t.$$

- $\delta < 2 \Rightarrow$ squared-Bessel process hits zero
- $\delta \geq 2 \Rightarrow$ squared-Bessel process does not hit zero

0 Extensions?

Two questions:

(i) How to extend a pssMp after hitting 0 in the recurrent regime **(R)** with an instantaneous entrance from zero?

(ii) How to start a pssMp from the origin in the transient regime **(T)**? Are there extensions $\{\mathbb{P}^z, z \geq 0\}$ with the Feller property so that in particular $\mathbb{P}^0 := \text{w-}\lim_{z \downarrow 0} \mathbb{P}^z$ exists in the Skorokhod topology?

People: Barczy, Bertoin, Caballero, Chaumont, Döring, Fitzsimmons, Kyprianou, Pardo, Rivero, Savov, ...

0 Extensions for pssMp

Theorem (R): (Fitzsimmons '06, Rivero '07)
In the recurrent regime the following statements are equivalent:

- $\exists$ unique recurrent self-similar Markov extension
- $\exists$ a self similar excursion measure with summable excursion length
- $\exists\ \lambda \in (0, \alpha)$ such that
$$\mathbb{E}[e^{\lambda \xi_1}] = 1.$$

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Theorem (T): (Chaumont, Kyprianou, Pardo, Rivero '12, Bertoin, Savov '11)
In the recurrent regime the following statements are equivalent:

- The weak limit $\mathbb{P}^0 = \lim_{x \downarrow 0} \mathbb{P}^x$ exists.
- ξ has stationary overshoots meaning that the weak limit of overshoots
$$O := \text{w-}\lim_{x \uparrow \infty}(\xi_{\tau_x} - x) \quad \text{exists,}$$
where τ_x is the first time ξ enters $[x, \infty)$.

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Note: As a consequence of Doney, Maller '02 the latter property can be characterised in terms of the Lévy triplet of ξ (integral conditions).

0 Real self-similar Markov processes

Aim: Similar results for transient regime for real self-similar Markov processes!

- What does convergence of overshoots mean in that context?
- Can one characterise the regime where overshoot distributions exist similar as in the classical case?

Real self-similar Markov process of index $\alpha > 0$:

- strong càdlàg Markov family $\{\mathbb{P}^z, z \in \mathbb{R}\backslash\{0\}\}$ on the state space $\mathbb{R}$ - 0 being an absorbing cemetery state - such that the scaling property holds:

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for all $z \neq 0$ and $c > 0$, where $Z = (Z_t)_{t\geq 0}$ denotes the canonical process.

The analogue of the Lamperti representation is based on Markov additive processes!

0 Markov additive processes

MAP: A càdlàg Markov process (ξ, J) is a MAP if J is a Markov chain with finite state space E and if there exist independent iid sequences

- $(\xi^{i,n})_{n\in\mathbb{N}_0}$ of Lévy processes for $i \in E$
- $(\Delta^n_{i,j})_{n\in\mathbb{N}}$ of real random variables for $i, j \in E$,

such that, if T_n is the nth jump-time of J, one has

$$\xi_t = \begin{cases} \xi_{T_n} + \xi^{J_{T_n},n}_{t-T_n} & : t \in (T_n, T_{n+1} \wedge \mathbf{k}), n \in \mathbb{N}_0 \\ \xi_{T_n-} + \Delta^n_{J_{T_n-},J_{T_n}}, & : t = T_n < \mathbf{k}, n \in \mathbb{N} \\ \xi_t = -\infty & : t \geq \mathbf{k} \end{cases}$$

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Overshoots: A MAP is said to have stationary overshoots if the limit

$$\text{w-}\lim_{a\to\infty} \mathbb{P}((\xi_{\tau_a^+} - a, J_{\tau_a^+}) \in \cdot \,|(\xi_0, J_0) = (0, i))$$

exists and does not depend on i. Here: τ_a^+ first entrance time into $[a, \infty) \times E$.

0 Lamperti representation

Lamperti-Kiu representation: (Kiu '80, Chaumont et al '13)

- $\exists$ a Markov additive process $(\xi, J) = (\xi_t, J_t)_{t\geq 0}$ on $\mathbb{R} \times \{\pm 1\}$ (possibly killed with cemetery state $-\infty$), such that, under $\mathbb{P}^z$ for $z \neq 0$,

$$Z_t = \exp(\xi_{A(t)}) J_{A(t)}, \qquad t \geq 0,$$

where (ξ, J) is started in $(\log(|z|), \mathrm{sgn}(z))$ and $A(\cdot)$ is as before.

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Focus: transient regime !

0 Main result

Theorem: (D, Döring, Kyprianou '15+) In the transient regime:

$\exists$ Feller extension $\{\mathbb{P}^z : z \in \mathbb{R}\} \iff$ the MAP has stationary overshoots.

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More explicitly: If the MAP has stationary overshoots, then there exists an extension $\{\mathbb{P}^z : z \in \mathbb{R}\}$ that is a strong càdlàg Markov family such that:

1. Under $\mathbb{P}^0$ the process leaves 0 instantaneously (continuously).
2. The corresponding transition semigroup (P_t) on $\mathbb{R}$ has the Feller property.
3. The family $\{\mathbb{P}^z : z \in \mathbb{R}\}$ is self similar.

Furthermore, $\mathbb{P}^0$ is the unique distribution satisfying one of the properties (1) or (2).

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Rem:

- The distribution $\mathbb{P}^0$ admits a potential theoretic interpretation.
- A characterisation of stationary overshoots is valid in the spirit of Doney and Maller.

0 Roadmap

Roadmap:

1. Definition of a candidate $\mathbb{P}^0$
2. Verification of
$$\mathbb{P}^0 = \text{w-}\lim_{x\to 0} \mathbb{P}^x$$
3. Characterisation of stationary overshoots

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ad 1: based on potential theory (Kuznestov measure, quasi-process)

ad 2: based on a technical lemma and fluctuation theory (potential measure of ascending ladder hight process)

ad 3: similar criterion as in Doney, Maller '02

We defer the discussion of one and start with steps two and three.

If time permits, we will state the theorem in the language of potential theory.

Verification of $\mathbb{P}^0 = \text{w-}\lim_{x\to 0} \mathbb{P}^x$

I Convergence lemma

Q: Once we defined a law $\mathbb{P}^0$, how can we verify that it is the right one?

Convergence lemma: Let $\{\mathbb{P}^z : z \in \mathbb{R}\backslash\{0\}\}$ be a strong càdlàg Markov family and $\mathbb{P}^0$ a law on the Skorokhod space. Suppose

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Then the mapping

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(1a) $\lim_{\varepsilon\to 0} \limsup_{|z|\to 0} \mathbb{E}^z[T_\varepsilon] = 0$

(1b) $\text{w-}\lim_{z\to 0} \mathbb{P}^z(Z_{T_\varepsilon} \in \cdot) =: \mu_\varepsilon(\cdot)$ exists for all $\varepsilon > 0$

(1c) $\mathbb{R}\backslash\{0\} \ni z \mapsto \mathbb{P}^z$ is continuous (weak topology on Skorokhod space)

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(2a) $\mathbb{P}^0$-almost surely, $Z_0 = 0$ and $Z_t \neq 0$ for all $t > 0$

(2b) $\mathbb{P}^0((Z_{T_\varepsilon + t})_{t \geq 0} \in \cdot) = \mathbb{P}^{\mu_\varepsilon}(\cdot)$ for every $\varepsilon > 0$

Then the mapping

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Verification of properties (1a)-(1c)

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The proof is based on fluctuation theory which can be developed for MAPs in analogy to the Lévy case.

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Using the Lamperti representation one obtains

$$\mathbb{E}^z[T_\varepsilon]\le\varepsilon^\alpha\sum_{j,k\in\{\pm 1\}}\frac{\pi_j}{\pi_{\mathrm{sgn}(z)}}\sum_{l\in\{\pm 1\}}\int_{[0,\infty)}e^{-\alpha y}\,\hat{U}^+_{j,l}(dy)\int_{[0,\log(\varepsilon/|z|)]}e^{-\alpha(\log(\varepsilon/|z|)-z)}\,U^+_{k,l}(dz)$$

in terms of the potential measure $U^+_{k,l}$ ($\hat{U}^+_{j,l}$) of the ascending (descending) Markov additive ladder height process of ξ.

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in terms of the potential measure $U^+_{k,l}$ ($\hat{U}^+_{j,l}$) of the ascending (descending) Markov additive ladder height process of ξ.

The integrals are finite and the key renewal theorem (for MAPs) yields convergence of the latter integral.

I Verification of properties (1a)-(1c)

(1b) w- $\lim_{z\to 0} \mathbb{P}^z(Z_{T_\varepsilon} \in \cdot) =: \mu_\varepsilon(\cdot)$ exists for all $\varepsilon > 0$

By the Lamperti-Kiu representation this is equivalent to the MAP having stationary overshoots.

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(1c) $\mathbb{R}\backslash\{0\} \ni z \mapsto \mathbb{P}^z$ is continuous (weak topology on Skorokhod space)

Consequence of the Lamperti-Kiu representation.

Criterion for stationary overshoots

II Criterion for stationary overshoots

Q: When does a MAP (ξ, J) has stationary overshoots?

Theorem: (D, Döring, Kyprianou '15+) The MAP has stationary overshoots, if ξ_1 has finite absolute moment and either of the following holds:

(i) (ξ, J) drifts to $+\infty$

(ii) (ξ, J) oscillates and

$$\int_1^\infty \frac{x\,\Pi([x,\infty))}{1+\int_0^x \int_y^\infty \Pi((-\infty,-z])\,dz\,dy}\,dx < \infty, \qquad \text{(TO)}$$

where Π is the measure

$$\Pi = \sum_{i \in E} \Pi_i + \sum_{\substack{i \to j \\ \text{pos. trans. of } J}} \mathcal{L}(\Delta_{i,j}),$$

and Π_i denotes the Lévy measure of $\xi^{i,n}$ from the Lamperti representation.

II Idea of the proof

One proves the following statements:

It suffices to characterise the case where (ξ, J) has tight overshoots (thanks to fluctuation theory).

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A close inspection of the property (TO) shows that it is quite robust: it is preserved by L^2-perturbations and behaves well for summands as they appear in one cycle from state i to i.

Construction of $\mathbb{P}^0$

III Kuznetsov measure

We use results from potential theory. To explain these let

- (P_t) be a Feller semigroup on a locally compact space E
- W the set of all functions $w : \mathbb{R} \to E \cup \{\partial\}$ (∂ denoting a cemetery state) such that there exist $\alpha(w) < \beta(w)$ with
  - w is E-valued and càdlàg on $(\alpha(w), \beta(w))$
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Kuznetsov measure: For a family of σ-finite measures $(\eta_t)_{t\in\mathbb{R}}$ on E with $\eta_s P_{t-s} \le \eta_t$ for $s < t$ (entrance rule) there exists a σ-finite measure $\mathcal{Q}_\eta$ on W such that for all $t_0 < \ldots < t_n$

$$\mathcal{Q}_\eta(\alpha(Y) < t_0, Y_{t_0} \in dx_0, \ldots, Y_{t_n} \in dx_n, t_n < \beta(Y)) \\ = \eta_{t_0}(dx_0) P_{t_1-t_0}(x_0, dx_1) \ldots P_{t_n-t_{n-1}}(x_{n-1}, dx_n).$$

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Application: For the MAP an invariant measure is given by $m(dx, i) = dx\, \pi_i$ and the Kuznetsov measure for $\eta_t \equiv m$ is denoted by $\mathcal{Q}_{\mathrm{MAP}}$. One has

$$\alpha = -\infty, \ \mathcal{Q}_{\mathrm{MAP}}\text{-a.e.}$$

III Lamperti-like time changes

Random time change: Suppose that $h : E \to (0,\infty)$ is locally bounded and measurable and set for $w \in W$ with $\int_{(\alpha,u]} h(w_s)\,ds < \infty$ for a $u > \alpha$

$$A_t = \Big(\int_{(\alpha,\cdot]} h(Y_s)\,ds\Big)^{-1}.$$

Time changed semigroup: $(\tilde{P}_t)$ given by

$$\tilde{P}_t f(x) = \mathbb{E}^x[f(Y_{A_t})],$$

where paths $(Y_t)_{t\geq 0}$ are interpreted as elements of $W_0 = \{\alpha = 0\} \subset W$.

III Construction of $\mathbb{P}^0$ via Kaspi '88

Thm: (Kaspi '88) Suppose that m is a (P_t)-invariant measure and that

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There is a Kuznetsov measure $\tilde{\mathcal{Q}}$ supported on $W_0 = \{\alpha = 0\}$ such that

$$\tilde{\mathcal{Q}}(\,\cdot \cap \{\beta > t\}) = \mathcal{Q}_m(\pi(Y) \in \,\cdot\,, \; 0 < A_t \leq 1),$$

where

$$\pi(Y)_t = \begin{cases} Y_{A_t}, & t > 0, \\ \partial, & t \leq 0. \end{cases}$$

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Application: Choose $h(x,i) = e^{\alpha x}$ and apply the theorem onto the MAP (restrict attention to the case where the MAP drifts to infinity)

- integral finite? (a $\mathcal{Q}_{\mathrm{MAP}}$-process backwards in time is adjoint MAP)
- $Y_t^1 \to -\infty$ as $t \downarrow -\infty$, $\mathcal{Q}_{\mathrm{MAP}}$-a.e. $\Rightarrow$ same true for $t \downarrow 0$, $\tilde{\mathcal{Q}}$-a.e.
- $\tilde{\mathcal{Q}}$ is a finite measure and its normalisation is $\mathbb{P}^0$ (tightness of overshoots)

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3⇒1:

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Q: How to recover Kaspi's result with the concepts from the previous slide?

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The Kuznetsov measure $\tilde{\mathcal{Q}} = \tilde{\mathcal{P}}_{\tilde{m}} \circ (\theta_\alpha)^{-1}$ agrees with the respective measure in Kaspi's theorem.

IV Conclusion/summary

- The correspondence between self-similar extensions and stationary overshoots prevails in the transient $\mathbb{R}$-valued case.
- There is a characterization of stationary overshoots for MAPs similar to the one in Doney and Maller.
- The measure $\mathbb{P}^0$ is a normalised quasi-process with birth time set to zero.

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- In the recurrent case one obtains analogous statement holds with the limit corresponding to the excursion measure.

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