Transition densities of isotropic unimodal Lévy processes (asymptotics and estimates)

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Joint works with W. Cygan, M. Ryznar and B. Trojan

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Introduction

Let X_t be an isotropic unimodal Lévy process that means that the Lévy measure is $\nu(|x|)dx$, where

$$\nu : (0, \infty) \mapsto [0, \infty) \quad \text{and } \nu - \text{non-increasing}.$$

Hence

$$\mathbb{E}e^{i\xi X_t} = \int_{\mathbb{R}^d} e^{i\xi z} p(t, z)dz + \mathbb{P}(X_t = 0) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

where the Lévy-Khintchine exponent is equal

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi z))\nu(|z|)\mathrm{d}z + \sigma|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

for some $\sigma \geq 0$.

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for some $\sigma \geq 0$.

Important property: $p(t, \cdot)$ is radial non-increasing function for every $t > 0$.

Examples

Let $\varphi(\lambda) = \sigma\lambda + \int_0^\infty (1 - e^{-\lambda s})\mu(ds)$ (a Bernstein function).

Then $\psi(\xi) = \varphi(|\xi)$ is the Lévy-Khinchine exponent of a subordinated Brownian motion.

Let $\alpha \in (0, 2]$

- $\psi(\xi) = |\xi|^\alpha$;
- $\psi(\xi) = (|\xi|^2 + 1)^{\alpha/2} - 1$;
- $\psi(\xi) = \left(\log(1 + |\xi|^2)\right)^{\alpha/2}$;

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- $\nu(r) = r^{-d-\alpha}\mathbf{1}_{(0,1)}(r)$.

Estimates

Example: $\psi(\xi) = |\xi|^\alpha$, $\alpha \in (0,2)$. We have

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$$t^{-d/\alpha} = \left[\left(\frac{1}{t}\right)^{1/\alpha}\right]^d = \left[\psi^{-1}\left(\frac{1}{t}\right)\right]^d.$$

And

$$\frac{t}{|x|^{d+\alpha}} = \frac{t\psi\left(\frac{1}{|x|}\right)}{|x|^d}.$$

Hence

$$p(t,x) \approx \left[\psi^{-1}\left(\frac{1}{t}\right)\right]^d \wedge \frac{t\psi\left(\frac{1}{|x|}\right)}{|x|^d}.$$

Known results

If we know a density ν and $\sigma = 0$ it is known (Chen, Kim, Kumagai) that if

$$c\lambda^{-d-\beta}\nu(r) \leq \nu(\lambda r) \leq C\lambda^{-d-\alpha}\nu(r), \quad r > 0,\ \lambda > 1$$

for some $0 < \beta \leq \alpha < 2$ the appropriate estimates hold.

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Since

$$\psi(x) = \int (1 - \cos(xz))\nu(|z|)\mathrm{d}z$$

we have

$$c\lambda^{\beta_1}\psi(r) \le \psi(\lambda r) \le C\lambda^{\beta_2}\psi(r).$$

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Theorem (Bogdan-TG- Ryznar (2014))

Assume that there are $\alpha, \beta \in (0,2)$ *such that*

$$c\lambda^\beta \psi(\xi) \le \psi(\lambda\xi) \le C\lambda^\alpha \psi(\xi), \quad \lambda \ge 1, |\xi| > R.$$

Then there is $a > 0$ *such that, for* t, $|x| < a$

$$p_t(x) \approx [\psi^{-1}(1/t)]^d \ \wedge \ \frac{t\psi(1/|x|)}{|x|^d}.$$

Strong ratio limit

Example:
For $\psi(\xi) = |\xi|^\alpha$ we have

$$p(t,x) = t^{-d/\alpha} p(1, xt^{-1/\alpha}) = t^{-d/\alpha} p\left(1, (t\psi(1/|x|))^{-1/\alpha}\right).$$

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$$\frac{p(t,x)}{p(t,0)} = \frac{p\left(1, (t\psi(1/|x|))^{-1/\alpha}\right)}{p(1,0)}$$

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Hence

$$\frac{p(t,x)}{p(t,0)} \longrightarrow 1, \quad \text{if } t\psi(1/|x|) \to \infty.$$

Strong ratio limit

Proposition (Cygan-TG-Trojan)

Assume that there exist $\beta, c > 0$ *such that*

$$\psi(\lambda\xi) \geq c\lambda^{\beta}\psi(\xi), \quad \lambda \geq 1, \xi \in \mathbb{R}^d.$$

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We have

$$p(t,0) = (2\pi)^{-d} \int e^{-t\psi(\xi)} \mathrm{d}\xi.$$

Regular variation

Definition

- $f \in \mathcal{R}_\alpha^\infty$ if

$$\frac{f(\lambda r)}{f(r)} \xrightarrow{r\to\infty} \lambda^\alpha, \quad \text{for all } \lambda > 0.$$

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- $f(r) = r^\alpha \in \mathcal{R}_\alpha^0 \cap \mathcal{R}_\alpha^\infty$, $\alpha \in \mathbb{R}$

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- $f(r) = r^\alpha \log^\gamma(1 + r^\beta) \in \mathcal{R}_{\alpha+\gamma\beta}^0 \cap \mathcal{R}_\alpha^\infty$, $\alpha, \gamma \in \mathbb{R}$, $\beta \geq 0$.

Proposition

- *If* $\psi \in \mathcal{R}_{\alpha}^{\infty}$ *for some* $\alpha > 0$ *then*

$$\frac{p(t,0)}{\left[\psi^{-1}\left(\frac{1}{t}\right)\right]^{d}} \xrightarrow{t\to 0^{+}} C(d,\alpha).$$

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Example:
For $\psi(\xi) = |\xi|^\alpha \log^\alpha(1+|\xi|^\alpha)$, $\alpha \in \left(0, \frac{2}{3}\right]$, we have

$$t^{d/\alpha} \log^d(t) p(t,0) \xrightarrow{t\to 0^+} C(d,\alpha).$$

Small $t\psi$

Pólya, Blumenthal-Getoor proved for $\psi(\xi) = |\xi|^\alpha$, $\alpha \in (0, 2)$,

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$$\infty \leftarrow \frac{|x|}{t^{1/\alpha}} = \left(\frac{|x|}{t}\right)^{1/\alpha} = \left(\frac{1}{t\psi\left(\frac{1}{|x|}\right)}\right)^{1/\alpha}.$$

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Equivalently $t\psi\left(\frac{1}{|x|}\right) \to 0^+$. Hence

$$\frac{|x|^d}{t\psi\left(\frac{1}{|x|}\right)} p(t,x) = \left(|x| t^{-1/\alpha}\right)^{d+\alpha} p\left(1, xt^{-1/\alpha}\right) \xrightarrow{t\psi\left(\frac{1}{|x|}\right)\to 0} \mathcal{A}(d,\alpha).$$

Small $t\psi$ and $\alpha \in (0,2)$

Theorem (Cygan-TG-Trojan)

- *If* $\psi \in \mathcal{R}_\alpha^\infty$, *for some* $\alpha \in (0,2)$ *then*

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Lemma

Let $\nu(r) = r^{-d} g(r^{-1})$ *and* $\alpha \in (0, 2)$.

- *If* $g \in \mathcal{R}_\alpha^\infty$, *then* $\psi \in \mathcal{R}_\alpha^\infty$.
- *If* $g \in \mathcal{R}_\alpha^0$, *then* $\psi \in \mathcal{R}_\alpha^0$.

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- *If* $g \in \mathcal{R}^{0}_{\alpha}$, *then* $\psi \in \mathcal{R}^{0}_{\alpha}$.

Example:

Let $\nu(r) = r^{-d-\alpha} \log^{\beta}(1 + e^{\beta} + r)$, for $\alpha \in (0, 2)$, $\beta \geq 0$. Then

$$\frac{p(t,x)}{t|x|^{-d-\alpha} \log^{\beta}|x|} \xrightarrow[|x|\to\infty]{t|x|^{-\alpha}\log^{\beta}|x|\to 0} 1$$

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Similar result holds also for $\mathcal{R}_\alpha^0$.

Question: what with $\alpha = 0$ and $\alpha = 2$?

For $\alpha = 2$ see paper by Mimica and Kim (preprint 2016).

Example

Let $\psi(\xi) = \log(1 + |\xi|^\beta)$, $\beta \in (0, 2]$. Then

$$p(t, x) \approx \frac{t|x|^{\beta t}}{|x|^d}, \quad \text{for } t,\ |x| \text{ small.}$$

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Observe

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Hence

$$p(t,x) \approx t\left(\psi\left(\frac{e}{|x|}\right) - \psi\left(\frac{1}{|x|}\right)\right)|x|^{-d} e^{-t\psi\left(\frac{1}{|x|}\right)}.$$

Slowly varying symbols, $\psi \in \mathcal{R}_0$

Definition

- $f \in \Pi_\ell^\infty$, for $\ell \in \mathcal{R}_0^\infty$ if

$$\frac{f(\lambda r) - f(r)}{\ell(r)} \xrightarrow{r\to\infty} \log \lambda, \quad \text{for all } \lambda > 0.$$

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For instance $\psi(\xi) = \log(1 + |\xi|^\beta)$, $\beta \in (0, 2]$. We have

$$\psi(\lambda r) - \psi(r) = \log\left(\frac{1 + (\lambda r)^\beta}{1 + r^\beta}\right) \xrightarrow{r\to\infty} \beta \log \lambda.$$

Hence $\psi \in \Pi_\ell^\infty$ with $\ell \equiv \beta$.

Asymptotics $\alpha = 0$

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- $$\frac{|x|^d \nu(|x|)}{\ell\left(\frac{1}{|x|}\right)} \xrightarrow{x\to 0} C > 0.$$

In particular $C = \frac{\Gamma(d/2)}{2\pi^{d/2}}$.

Similar result holds also for Π_ℓ^0.

Recall that for $\psi(\xi) = \log(1 + |\xi|^\beta)$, $\beta \in (0, 2]$ we have

$$p(t,x) \approx \frac{t|x|^{\beta t}}{|x|^d}, \quad \text{for } t,\ |x| \text{ small.}$$

By the above theorem

$$p(t,x) \sim C\frac{t}{|x|^d}, \quad x \to 0 \text{ and } |x|^{\beta t} \to 1.$$

Observe that $p(t, 0) = \infty$ and $e^{-t\psi} \notin L^1$.

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Remark: If $\ell \to \infty$ at infinity then $p(t,0) < \infty$.

Estimates

Let $\psi \in \Pi_\ell^\infty$ for $\ell \in \mathcal{R}_0^\infty$ and ℓ be bounded. We have

$$p(t,x) \approx t|x|^{-d}\ell\left(\frac{1}{|x|}\right), \quad |x|,\ t\psi\left(\frac{1}{|x|}\right) < \varepsilon.$$

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$$p(t,x) \approx t|x|^{-d}\ell\left(\frac{1}{|x|}\right)e^{-t\psi\left(\frac{1}{|x|}\right)}, \quad t,\ \left(t\psi\left(\frac{1}{|x|}\right)\right)^{-1} < \varepsilon.$$

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Proposition (TG-Ryznar-Trojan)

There exist c, C such that for any non-increasing ν

$$p(t,x) \geq Ct\nu(|x|)e^{-ct\psi\left(\frac{1}{|x|}\right)}, \quad t > 0,\ x \in \mathbb{R}^d.$$

Lemma (TG-Ryznar-Trojan)

Let $\psi \in \Pi_\ell^\infty$ *for* $\ell \in \mathcal{R}_0^\infty$ *and* ℓ *be bounded. Then there is* $C > 0$ *such that,*

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Theorem (TG-Ryznar-Trojan)

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2	???	$p(t,0) \wedge (*)$

where

$$(*) = t\left(\psi(|x|^{-1} - (2|x|)^{-1}\psi'(|x|^{-1})\right)|x|^{-d} + p(t,0)e^{-c(|x|\psi^{-1}(1/t))^2}.$$

Bounded ψ

We have

$$p(t,x) = e^{-tN(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{t^k}{k!} N^{*k}(x)$$

$$= e^{-tN(\mathbb{R}^d)} t\nu(|x|) \left(1 + t \sum_{n=0}^{\infty} \frac{t^n}{(n+2)!} \frac{N^{*(n+2)}(x)}{\nu(|x|)} \right).$$

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Hence, for t and $|x|$ small we have

$$p(t,x) \sim t\nu(|x|).$$

Example

Let $\psi(\xi) = \log(1 + |\xi|^\alpha)$, $\alpha \in (0, 2)$.

Then, for $t < 3d/\alpha$,

$$p(t,x) \approx \begin{cases} t|x|^{-d-\alpha}, & |x| \geq 1, \\ t \min\left\{ \log(2|x|^{-\alpha}), (t - d/\alpha)^{-1} \right\}, & |x| < 1,\ t > d/\alpha, \\ t(\log(2|x|^{-\alpha}) + |x|^{\alpha t - d}), & |x| < 1,\ t \in (0, d/\alpha]. \end{cases}$$

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And for $t \geq 3d/\alpha$,

$$p(t,x) \approx \min\left\{t^{-d/\alpha}, t|x|^{-d-\alpha}\right\}.$$

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Green function

Let

$$G(x) = \int_0^\infty p(t,x)dt.$$

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Theorem

Suppose $d \geq 6$. Then

$$\lim_{x\to 0} \frac{G(x)}{|x|^{-d}\psi(|x|^{-1})^{-1}} = c > 0, \qquad (1)$$

if and only if $\psi \in \mathcal{R}_\alpha^\infty$, for some $\alpha > 0$. In particular, (1) implies that

$$c = 2^{-\alpha}\pi^{d/2}\frac{\Gamma\left((d-\alpha)/2\right)}{\Gamma\left(\alpha/2\right)}.$$

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Let

$$U_t(r) = \mathbb{P}\left(0 < |X_t| \leq \sqrt{r}\right) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{\sqrt{r}} u^{d-1} p(t,u)\,\mathrm{d}u.$$

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We observe that by the Tonelli's theorem

$$\begin{aligned}
\lambda \mathcal{L} U_t(\lambda) &= \int_{\mathbb{R}^d} e^{-\lambda|x|^2} p(t,x)\,\mathrm{d}x. \\
&= (4\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-t\psi(\xi\sqrt{\lambda})} e^{-|\xi|^2/4}\,\mathrm{d}\xi - \mathbb{P}\big(|X_t| = 0\big) \\
&= \frac{2^{1-d}}{\Gamma(d/2)} \int_0^{\infty} e^{-t\psi(r\sqrt{\lambda})} e^{-r^2/4} r^{d-1}\,\mathrm{d}r - \mathbb{P}\big(|X_t| = 0\big)
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Let

$$Q_t(r) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^{\sqrt{r}} u^{d+1} p(t,u)\,\mathrm{d}u,$$

Then

$$\mathcal{L}\{\mathrm{d}U_t\}(s^{-1}) = \int_0^s \mathcal{L}\{\mathrm{d}Q_t\}(r^{-1}) r^{-2}\,\mathrm{d}r.$$

Theorem (TG-Ryznar-Trojan)

Let $\{Q_t : t \geq 0\}$ *be a family of non-decreasing and non-negative functions on* $[0, \infty)$ *such that there are two families of positive functions* $\{q_t : t \geq 0\}$ *and* $\{w_t : t \geq 0\}$ *satisfying*

$$\frac{\lambda \mathcal{L}\{\mathrm{d}Q_t\}(\lambda)}{q_t(\lambda)} \xrightarrow[\lambda \to \infty]{w_t(\lambda) \to 0} 1.$$

We assume that there is $\rho \geq 0$*,*

$$\frac{q_t(\lambda x)}{q_t(x)} \xrightarrow[x \to \infty]{w_t(x) \to 0} \lambda^{\rho}, \quad \lambda > 0.$$

+ some additional conditions

Then

$$\frac{Q_t(r)}{r q_t(r^{-1})} \xrightarrow[r \to 0]{w_t(r^{-1}) \to 0} \frac{1}{\Gamma(\rho + 2)}.$$

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- $\psi \in \mathcal{R}_\alpha^\infty$, *for some* $\alpha \in (0,2)$.
- $$\frac{|x|^d p(t,x)}{t\psi\left(\frac{1}{|x|}\right)} \xrightarrow[x\to 0]{t\psi\left(\frac{1}{|x|}\right)\to 0} C > 0.$$
- $$\frac{|x|^d \nu(|x|)}{\psi\left(\frac{1}{|x|}\right)} \xrightarrow{x\to 0} C > 0.$$

Observe

$$\psi(r) = \int_0^\infty k(\rho^{-1} r)\rho^{-d}\nu\big(\rho^{-1}\big)\rho^{-1}\,\mathrm{d}\rho$$

where σ denotes the spherical measure on the unite sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, and

$$k(r) = \int_{\mathbb{S}^{d-1}} \big(1 - \cos(r\langle u_0, u\rangle)\big)\sigma(\mathrm{d}u).$$

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Let us recall the definition of the *Mellin convolution*. For $f, g : [0,\infty) \to \mathbb{C}$

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Next we checked assumptions of Drasin-Shea Theorem and applied it to get the claim.

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Thank you.