Fluctuation theory for Markov Additive processes and applications to self-similar processes

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[1] based on work in progress with Loïc Chaumont, Andreas Kyprianou and Bati Sengul

Consider a completed, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), (\mathbb{P}_{\varphi,x}, (\varphi, x) \in E \times \mathbb{R}))$, where E is a locally compact space with a countable base, Δ is some isolated state and $E \cup \{\Delta\}$ endowed with its Borel σ-field.

Definition (Neveu (1961), Çinlar (1972))

A *Markov additive process (MAP)* is an $E \times \mathbb{R}$-valued strong Markov process $\{((J, \xi), \mathbb{P}_{\varphi,x}\}$ with cemetery state (Δ, ∞), lifetime ζ, such that

(i) the paths of (J, ξ) are right continuous on $[0, \infty)$, have left-limits and are quasi-left continuous on $[0, \zeta)$;

(ii) J is a strong Markov process;

(iii) for any $(\varphi, z) \in E \times \mathbb{R}$, $t, s \geq 0$ and $f : E \times \mathbb{R} \to \mathbb{R}$ measurable and positive

$$\mathbb{P}_{\varphi,z}\left(f(J_{t+s}, \xi_{t+s} - \xi_t), t + s < \zeta | \mathcal{F}_t\right) = \mathbb{P}_{J_t,0}\left(f(J_s, \xi_s), s < \zeta\right) 1_{\{t<\zeta\}}.$$

Motivation: Lamperti's transform

Theorem (Lamperti (1972), Kiu (1980), Chaumont, Pantí R. (2013), Kuznetsov, Kyprianou, Pardo, Watson (2012), Alili, Chaumont, Graczyk, and Zak (2016))

Let X be a $\mathbb{R}^d$ valued strong Markov process having càdlàg paths, quasi-left continuous, and that has the scaling property: there exists an $\alpha > 0$ such that for any $c > 0$ the process

$$\{(cX_{c^{-\alpha}t},\ t \geq 0), \mathbb{P}_x\} \stackrel{Law}{=} \{(X_t,\ t \geq 0), \mathbb{P}_{cx}\}, \qquad x \in \mathbb{R}^d.$$

Assume X dies at its first hitting time of 0. The process (J, ξ) defined by

$$J_t = \frac{X_{\tau(t)}}{|X_{\tau(t)}|}, \qquad \xi_t = \log(|X_{\tau(t)}|/|X_0|), \qquad t \geq 0,$$

with

$$\tau(t) = \inf\{s > 0 : \int_0^s |X_u|^{-\alpha} du > t\}, \qquad t > 0,$$

and $(J, \xi)_{\tau(t)} = (\Delta, \infty)$ if $\tau(t) = \infty$, is a $\mathbb{S}^{d-1} \times \mathbb{R}$ valued Markov additive process.

- What can be said about $\mathbb{S}^{d-1} \times \mathbb{R}$-valued MAPs?
- How do we characterise the MAP who is behind a stable process or a transformation of it?
- Is there a fluctuation theory that allow us to describe the (J, ξ) from its extrema?
- If yes, can this be used to get a better understanding of stable processes?

Since the 70's many prominent authors contributed to the study and development of applications of MAPs, mainly with finite and countable state space, but also in general: *Arjas, Asmussen, Boxma, Çinlar, Grigelionis, Iscoe, Ivanovs, Kaspi, Kella, Kyprianou, Ney, Nummelin, Maisonneuve, Palmowski, Pistorius, Prabhu, Speed... many others.*

MAPs in countable state space have applications in queueing, risk theory, financial mathematics, self-similar Markov processes theory, statistical physics...

MAPs in more general state spaces are less popular due to its technicalities but they are relevant at least in the study of excursions from a set, as shown by Çinlar and Kaspi (1982), and self-similar Markov processes in $\mathbb{R}^d$, and in particular for stable processes.

E countable

When E is finite or countable the process J is a Markov chain that describes the phases of the process and ξ evolves as a concatenation of independent Lévy processes $(\xi^i, i \in E)$ shifted by an independent sequence of r.v. $(U^n_{i,j}, (i,j) \in E \times E, n \geq 1)$.

- J starts in state j, ξ moves as ξ^j,
- at an exponential time T_1 of parameter q_j, J jumps to a new position, say k, with probability $p_{j,k}$, and stays there for an exponential time of parameter q_k,
- at time T_1, ξ jumps from position ξ_{T_1-} to position $\xi_{T_1-} + U^1_{j,k}$ and from there it evolves as ξ^k
- and so on

Notice ξ has jumps coming from each Lévy process and from J. Conditionally on J, ξ has jumps at the fixed times $(T_n, n \geq 1)$.

E finite

When E is finite, the dynamics of (J, ξ) are determined by:

- the infinitesimal generator of J, say $Q = (q_{i,j}, i, j \in E)$, $(q_{i,i} = 0, i \in E)$.
- the laws of $(U_{i,j}, i, j \in E)$ say $F_{i,j}(dy) = \mathbb{P}(U_{i,j} \in dy)$,
- the characteristic exponents of $(\xi^j, j \in E)$, say $(\Psi_j, j \in E)$ $\mathbb{E}\left(\exp\{\lambda \xi^j_t\}\right) = \exp\{t\Psi_j(\lambda)\}$,

The transition semigroup of (J, ξ) is characterised through its matrix exponent $K(\lambda) = (K_{i,j}(\lambda), i, j \in E)$ as

$$F^{(t)}_{i,j}(\lambda) = \mathbb{E}_i\left(\exp\{\lambda \xi_t\} 1_{\{J_t = j\}}\right) = \exp\{tK(\lambda)\}_{i,j}, \qquad i, j \in E, t \geq 0,$$

where

$$K(\lambda) = Q + (\Psi_j(\lambda))_{\text{diag}} + \left(q_{i,j}\left(\mathbb{E}\left(\exp\{\lambda U_{i,j}\}\right) - 1\right)\right)_{i,j \in E}.$$

See Asmussen's book *Applied Probability and Queues*.

In general, conditional to the driving process J

- ξ has independent increments,
- $\xi_t = A_t + \xi_t^f + \xi_t^c + \xi_t^d, \qquad t \geq 0,$
  where $\sigma(\xi_t^F, t \geq 0)$, $\sigma(\xi_t^d, t \geq 0)$ and $\sigma(\xi_t^c, t \geq 0)$ are conditionally independent, and (J, A) and $(J, \xi^{f/c/d})$ are MAPS.
- A is continuous additive functional of J
- Y^c is a Gaussian process,
- Y^f is a purely discontinuous process with discontinuities fixed by J,
- Y^d is a stochastically continuous process with independent increments and does not jumps at the same time as J.

A Lévy-Khintchine formula

Under rather general assumptions

$$
\begin{aligned}
&\mathbb{E}_\varphi\left[\exp\{i\lambda\xi_t\}|J\right] \\
&= \left\{\prod_{s\le t} F^\lambda_{\substack{J_{s-},J_s \\ J_{s-}\neq J_s}}(\omega)\right\}\exp\left\{i\lambda A_t - \frac{1}{2}\lambda^2 C_t.\right\} \\
&\times \exp\left\{\int_0^t d\widetilde{H}_s \int_{\mathbb{R}} \widetilde{\Pi}_s(J_s,dy)\left(e^{i\lambda y} - 1 - \frac{i\lambda y}{1+|y|^2}\right)\right\}
\end{aligned}
\tag{1}
$$

where

- **a)** for each (φ_0,φ_1), $F^\lambda_{\varphi_0,\varphi_1}$ is a characteristic function in λ and is $\mathcal{K}$-measurable for fixed λ.
- **b)** $A=(A_t, t\ge 0)$ is an additive functional of X.
- **c)** $C=(C_t, t\ge 0)$ is a non-negative continuous additive functional of X.
- **d)** for each j, $\widetilde{\Pi}_s(j,dy)$ is a Lévy measure and it is J-measurable for fixed λ, H is a continuous additive functional.

A Lévy-Khintchine formula

Under rather general assumptions

$$\begin{aligned} & \mathbb{E}_{\varphi}\left[\exp\{i\lambda\xi_t\}|J\right] \\ & = \left\{\prod_{\substack{s\leq t \\ J_{s-}\neq J_s}} F^{\lambda}_{J_{s-},J_s}(\omega)\right\} \exp\left\{i\lambda A_t - \frac{1}{2}\lambda^2 C_t.\right\} \\ & \times \exp\left\{\int_0^t d\widetilde{H}_s \int_{\mathbb{R}} \widetilde{\Pi}_s(J_s, dy)\left(e^{i\lambda y} - 1 - \frac{i\lambda y}{1+|y|^2}\right)\right\} \end{aligned} \quad (1)$$

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a) for each (φ_0, φ_1), $F^{\lambda}_{\varphi_0,\varphi_1}$ is a characteristic function in λ and is $\mathcal{K}$-measurable for fixed λ.

b) $A = (A_t, t \geq 0)$ is an additive functional of X.

c) $C = (C_t, t \geq 0)$ is a non-negative continuous additive functional of X.

d) for each j, $\widetilde{\Pi}_s(j, dy)$ is a Lévy measure and it is J-measurable for fixed λ, H is a continuous additive functional.

Can we describe $(H, \widetilde{\Pi})$ for the MAP behind the stable?

Lévy systems (compensation formula)

Lemma (Çinlar, (1975), after Beneviste and Jacod (1973))

There exists a continuous additive functional H and a kernel Π from E to $E \times \mathbb{R}$ such that for every f measurable and positive

$$\mathbb{E}_j\left(\sum_{s\le t} F(J_{s-}, J_s, \xi_{s-}, \xi_s) 1_{\{J_{s-}\neq J_s \text{ or } \xi_{s-}\neq \xi_s\}}\right) = \mathbb{E}_j\left(\int_0^t dH_s \int_{E\times\mathbb{R}} L(J_s, d\varphi, dy) F(J_s, \varphi, \xi_s, \xi_s + y)\right), \tag{2}$$

Moreover,

$$\Pi(j, d\varphi, dy) = 1_{\{\varphi = j\}}\widetilde{\Pi}(j, dy) + 1_{\{\varphi\neq j\}} K(j, d\varphi) F_{j,\varphi}(dy).$$

The Lévy kernel of the MAP associated to a Stable processes

Assume X is a rotationally invariant stable process in $\mathbb{R}^n$, its Lévy measure admits the polar representation $\Pi_X(dy) = \sigma(ds)\frac{dr}{|r|^{1+\alpha}}1_{\{sr\in dy\}}$, where σ is the uniform measure on $\mathbb{S}^{n-1}$.

Let $f : \mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1} \to \mathbb{R}$ be any positive measurable test function such that $f(\cdot, 0, \cdot) = 0$, and consider the expression ,

$$\mathbb{E}_{0,j}\left(\sum_{s>0} f(s, \xi_s - \xi_{s-}, J_s - J_{s-})\right)$$
$$= \mathbb{E}_{0,\frac{X_0}{|X_0|}}\left(\sum_{s>0} f\left(\int_0^s |X_u|^{-\alpha}du, \log\left(|X_s|/|X_{s-}|\right), \frac{X_s}{|X_s|} - \frac{X_{s-}}{|X_{s-}|}\right)\right)$$
$$= \mathbb{E}_{0,\frac{X_0}{|X_0|}}\left(\int_0^\infty dv \int_{J_v+\theta\in\mathbb{S}^{n-1}} \sigma(J_v + d\theta)\right.$$
$$\left.\times \int_{x\in(-\infty,\infty)\setminus\{0\}} dx \frac{e^{nx}}{|(e^x-1)^2 - 2e^x < \theta, J_v > |^{(\alpha+n)/2}} f(v, x, \theta)\right).$$

Fluctuation Theory

The fluctuation theory of MAPs describes the paths of (J, ξ) seen from the past supremum (infimum) and as its counterpart for Lévy processes has many important applications.

Lemma (Kaspi (1992))

Let $I_t = \inf_{s \le t} \xi_s$, $t \ge 0$, *and the reflected process* $U_t := \xi_t - I_t, t \ge 0$. *The process* $((J_t, \xi_t, U_t), t \ge 0)$ *is a standard Markov process.*

Proof.

Same proof as in the Lévy case. □

Duality

(Duality) Assume there is a measure π s.t. $\{(J,\xi)_t, t \geq 0\}$ is in weak duality with $\{(J,-\xi)_t, t \geq 0\}$ w.r.t. $dx\pi(d\theta)$, i.e.

$$\int_{E\times\mathbb{R}} \pi(d\varphi)dx f(\varphi,x)\,\mathbb{E}_{\varphi,x}\left(g(J_t,\xi_t)\right) = \int_{E\times\mathbb{R}} \pi(d\varphi)dx g(\varphi,x)\,\mathbb{E}_{\varphi,x}\left(f(J_t,-\xi_t)\right)$$

Lemma

Let $t > 0$ fixed. The process $\{(J_{(t-s)-}, \xi_{t-} - \xi_{(t-s)-}), 0 \leq s \leq t\}$ under $\mathbb{P}_{\pi,0}$ has the same law as $\{(J,\xi)_s, 0 \leq s \leq t\}$ under $\mathbb{P}_{\pi,0}$.

Lemma (Kaspi (1982), Palmowski et al. (2011), Ivanovs (2015) Kyprianou et al. (2016))

Assume that the duality condition is satisfied and that E is finite. Let $\mathbf{e}_q$ an independent exponential r.v. of parameter q, $\underline{G}_q = \sup\{s < \mathbf{e}_q : \xi_s = \underline{\xi}_s\}$.

(i) *The pairs of random variables $(\underline{G}_q, \underline{\xi}_{\mathbf{e}_q})$ and $(\mathbf{e}_q - \underline{G}_q, \xi_{\mathbf{e}_q} - \underline{\xi}_{\mathbf{e}_q})$ are conditionally independent given $(J_{\underline{G}_q -}, J_{\underline{G}_q})$.*

(ii) *The conditional laws are characterised through the q-potentials of the excursion measures from the supremum and the infimum, respectively.*

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Yes, by means of the theory of exit systems of Maissoneuve.

Definition

The downward ladder set

$$\underline{M} = \{t : (J_t, \xi_t, \xi_t - \inf_{s \le t} \xi_s) \in E \times \mathbb{R} \times \{0\}\}.$$

$$R = \inf\{t > 0 : t \in \underline{M}^{\mathsf{cl}}\}.$$

The set of regular points

$$\widetilde{F} = \{j \in E : \mathbb{P}_{j,x,0}(R = 0) = 1, \forall x \in \mathbb{R}\}.$$

For simplicity we often assume that $\widetilde{F} = E$.
Albeit this assumption may easily fail the results we will describe are true in greater generality.
Take $E = \{0, 1\}$, ξ^0 a stable process and ξ^1 a compound Poisson Process.

Exit system for the excursions from 0 for the process reflected in the infimum

Lemma (Maissoneuve (1982), Çinlar and Kaspi (1982), Kaspi (1983))

There exists an additive functional $\underline{\mathcal{L}}$, *with* 1*-potential smaller than* 1, *carried by* $E \times \mathbb{R} \times 0$ *and a kernel* P^* *from* $(E \times \mathbb{R} \times \mathbb{R}, \mathcal{E} \times \mathcal{R} \times \mathcal{R})$ *into* $(\Omega, \mathcal{F})$ *such that*

- $P^*_{\varphi,x,y}\left(1 - e^{-R}\right) \leq 1$ *for all* (φ, x, y)
- $\mathbb{E}.\left(\sum_{g \in G} Z_g H \circ \theta_g\right) = \mathbb{E}.\left(\int_0^\infty d\underline{\mathcal{L}}_s Z_s P^*_{J_s,\xi_s,U_s}(H)\right)$, *where* Z *is any positive predictable process,* H *any measurable and bounded functional and* G *is the set of left end points of intervals that are contiguous to* M.

Exit system for the excursions from 0 for the process reflected in the infimum

Lemma (Maissoneuve (1982), Çinlar and Kaspi (1982), Kaspi (1983))

There exists an additive functional $\underline{\mathcal{L}}$, with 1-potential smaller than 1, carried by $E \times \mathbb{R} \times 0$ and a kernel P^* from $(E \times \mathbb{R} \times \mathbb{R}, \mathcal{E} \times \mathcal{R} \times \mathcal{R})$ into $(\Omega, \mathcal{F})$ such that

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The excursion measure $\underline{N}_j$ is the image measure of $P^*_{j,0,0}$ under the mapping that stops the path at the end of the first excursion: $(J, \xi, \xi - I)_{\cdot \wedge R}$.

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- $P^*_{\varphi,x,y}\left(1 - e^{-R}\right) \leq 1$ *for all* (φ, x, y)
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The excursion measure $\underline{N}_j$ *is the image measure of* $P^*_{j,0,0}$ *under the mapping that stops the path at the end of the first excursion:* $(J, \xi, \xi - I)_{\cdot \wedge R}$.
Under $\underline{N}_j$ *the excursion process has the Markov property with the same semigroup as* (J, ξ) *killed when it passes below zero.*

The downward ladder process

Lemma (Kaspi (1982))

Let $\underline{\tau}_t = \inf\{s > 0 : \underline{\mathcal{L}}_s > t\}$, $t > 0$, *be the inverse local time process,* $\underline{H}_t = -\xi_{\underline{\tau}_t}$. *The downward ladder process*

$$\left\{(J_{\underline{\tau}_t}, \underline{\tau}_t, \underline{H}_t),\ t \geq 0\right\},$$

is Markov additive process with values in $E \cup \{\Delta\} \times [0,\infty] \times [0,\infty]$. *It is characterised through its Laplace transform:* $\exists\ \underline{\ell}, \underline{a} : E \to [0,\infty)$, *bounded s.t.*

$$\mathbb{E}_j\left(\exp\{-\lambda\underline{\tau}_t - \beta\underline{H}_t\}|J\right)$$
$$= \exp\left\{-\lambda\int_0^t \underline{\ell}(J_{\underline{\tau}_u})du - \beta\int_0^t \underline{a}(J_{\underline{\tau}_s})ds - \int_0^t du \underline{N}_{J_{\underline{\tau}_u}}(1 - \exp\{-\lambda\zeta - \beta\xi_\zeta\})\right\},$$

for all $\lambda \geq 0$.

We have the following analogue of Vigon's formula

$$\underline{N}_j\left(f(J_{\zeta-}, \xi_{\zeta-}, \zeta, J_\zeta, \xi_\zeta)1_{\{\xi_{\zeta-}\neq\xi_\zeta\}}\right)$$
$$= \underline{N}_j\left(\int_0^\zeta dH_t \int_{[-\infty,0)} \Pi(J_t, d\phi, dy) f(J_t, \xi_t, t, \phi, \xi_t + y)1_{\{y+\xi_t<0\}}\right).$$

Theorem (Time of maximum and last exit formula: I)

Let $U_j^-(d\varphi, dr, dz)$ *be the potential measure of the downward ladder process*
$U_j^-(d\varphi, dr, dz) = \mathbb{E}_j\left(\int_0^\infty ds 1_{\{J_{\underline{\tau}_s} \in d\varphi, \underline{\tau}_s \in dr, \underline{H}_s \in dz, \underline{\tau}_s < \infty\}}\right)$,
$\varphi \in E, r, z \in [0,\infty)$, *and* $\underline{G}_t = \sup\{s < t : \xi_s = \underline{\xi}_s\}$.

(i) *For any* $f : E \times \mathbb{R} \to \mathbb{R}^+$ *measurable*

$$\mathbb{E}_{j,x}\left(f(J_t, \xi_t) 1_{\{\underline{G}_t < t\}}\right)$$
$$= \int_{E\times[0,t]\times\mathbb{R}^+} U_j^-(d\varphi, dr, dz) \underline{N}_\varphi\left(f(J_{t-r}, x - z + \xi_{t-r}) 1_{\{t-r<\zeta\}}\right)$$

(ii) *Let* $\tau_0^- = \inf\{t > 0 : \xi_t < 0\}$.

$$\mathbb{E}_{j,x}\left(f(J_t, \xi_t) 1_{\{\underline{G}_t < t\}} 1_{\{t<\tau_0^-\}}\right)$$
$$= \int_{E\times[0,t]\times[0,x)} U_j^-(d\varphi, dr, dz) \underline{N}_\varphi\left(f(J_{t-r}, x - z + \xi_{t-r}) 1_{\{t-r<\zeta\}}\right)$$

Idea of Proof

Let $t > 0$, $\underline{G}_t = \sup\{s < t : \xi_s = \underline{\xi}_s\}$.

$$\begin{aligned}
\mathbb{E}_{j,x}\left(f(J_t,\xi_t)\right) &= \mathbb{E}_j\left(f(J_t, x+\xi_t)1_{\{t\in M\}}\right) \\
&\quad + \mathbb{E}_j\left(f(J_t, x+\underline{\xi}_{\underline{G}_t} + \xi_t - \underline{\xi}_{\underline{G}_t})1_{\{\underline{G}_t<t\}}\right) \\
&= T_t^c f(j) \\
&\quad + \mathbb{E}_j\left(\sum_{s\in G\cap[0,t)} 1_{\{R>t-s\}} f(J_{t-s}, x+\underline{\xi}_s + \xi_s - \underline{\xi}_s)\circ\theta_s\right) \\
&= T_t^c f(j) + \mathbb{E}_j\left(\int_{[0,t]} d\underline{\mathcal{L}}_s \underline{N}_{J_s}\left(f(J_{t-s}, x+\xi_s+\widetilde{\xi}_{t-s})1_{\{t-s<\zeta\}}\right)\right) \\
&= T_t^c f(j) + \int_{E\times[0,t]\times\mathbb{R}^+} U_j^-(d\varphi, dr, dz)\underline{N}_\varphi\left(f(J_{t-r}, x-z+\xi_{t-r})1_{\{t-r<\zeta\}}\right)
\end{aligned}$$

with $U_j^-(d\varphi, dr, dz) = \mathbb{E}_j\left(\int_0^\infty dt 1_{\{J_{\underline{\tau}_s}\in d\varphi, \underline{\tau}_s\in dr, \underline{H}_s\in dz, \underline{\tau}_s<\infty\}}\right)$

Theorem (Time of maximum and last exit formula: II)

The potential measure of the upward ladder process

$$U_j^-(d\varphi, dr, dz) = \mathbb{E}_j\left(\int_0^\infty ds 1_{\{J_{\underline{\tau}_s}\in d\varphi, \underline{\tau}_s \in dr, \underline{H}_s \in dz, \underline{\tau}_s<\infty\}}\right),$$

$\varphi \in E, r, z \in [0,\infty)$, *satisfies*

- **(ii)** *For each* $j \in E$ *there exists a kernel* $\underline{V}_j(t, d\varphi, dz)$ *from* $\mathbb{R}^+$ *to* $E \times \mathbb{R}^+$ *s.t.* $U_j^-(d\varphi, ds, dz)\underline{\ell}(\varphi) = \underline{V}_j(s, d\varphi, dz)ds$
- **(iii)** *If* $\underline{\ell} \equiv 0$ *then for all* $j \in E$, *and* $t > 0$, $\mathbb{P}_{j,x}(t \in \underline{M}) = 0$.
- **(iv)** *For any* $f : E \times \mathbb{R} \to \mathbb{R}^+$ *measurable, we have for almost all* $t > 0$,

$$\mathbb{E}_{j,x}\left(f(J_t, \xi_t)1_{\{\underline{G}_t=t\}}\right) = \int_{E\times[0,\infty)} \frac{U_j^-(d\varphi, ds, dz)}{ds}|_{\{s=t\}}\underline{\ell}(\varphi)f(\varphi, t, x-z)$$
$$= \int_{E\times[0,\infty)} \underline{V}_j(t, d\varphi, dz)\underline{\ell}(\varphi)f(\varphi, t, x-z)$$

- **(v)** $\mathbb{E}_{j,x}\left(f(J_t, \xi_t)1_{\{\underline{G}_t=t\}}1_{\{t<\tau_0^-\}}\right) = \int_{E\times[0,x)} \underline{V}_j(t, d\varphi, dz)\underline{\ell}(\varphi)f(\varphi, t, x-z)$

This is an extension of a result by Çinlar (1976) and Maissoneuve (1977).

Theorem

Assume that the duality condition is satisfied. Let $\mathbf{e}_q$ *an independent exponential r.v. of parameter* λ, $\underline{G}_\lambda = \sup\{s < \mathbf{e}_\lambda : \xi_s = \underline{\xi}_s\}$. *The pairs of random variables* $(\underline{G}_\lambda, \underline{\xi}_{\mathbf{e}_\lambda})$ *and* $(\mathbf{e}_\lambda - \underline{G}_\lambda, \xi_{\mathbf{e}_\lambda} - \underline{\xi}_{\mathbf{e}_\lambda})$ *are conditionally independent given* $(J_{\underline{G}_\lambda -}, J_{\underline{G}_\lambda})$.

$$\mathbb{E}_\pi\left(F(\underline{G}_{\mathbf{e}_\lambda}, \inf_{s \le \underline{G}_{\mathbf{e}_\lambda}} \xi_s) | J_{\underline{G}_{\mathbf{e}_\lambda}-} = j\right) =: \mathcal{W}_\lambda^+ F(j)$$

$$= \frac{\overline{\ell}(j)F(0,0) + \overline{N}_j\left(\int_0^\zeta du e^{-\lambda u} F(u, \epsilon_u) 1_{\{u<\zeta\}}\right)}{\overline{\ell}(j) + \frac{1}{\lambda}\overline{N}_j\left(1 - \exp\{-\lambda\zeta\}\right)},$$

$$\mathbb{E}_\pi\left(H(\mathbf{e}_\lambda - \underline{G}_{\mathbf{e}_\lambda}, \xi_{\mathbf{e}_\lambda} - \inf_{s \le \underline{G}_{\mathbf{e}_\lambda}} \xi_s) | J_{\underline{G}_{\mathbf{e}_\lambda}} = j\right) =: \mathcal{W}_\lambda^- H(j)$$

$$= \frac{\underline{\ell}(j)H(0,0) + \lambda\underline{N}_j\left(\int_0^\zeta du e^{-\lambda u} H(u, \epsilon_u) 1_{\{u<\zeta\}}\right)}{\underline{\ell}(j) + \frac{1}{\lambda}\underline{N}_j\left(1 - \exp\{-\lambda\zeta\}\right)}.$$

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Asymptotic behaviour of the marginals of the downward ladder measure is possible thanks to the Markov renewal theorem.

Proposition

The potential densities are given by the following. $\alpha \in (0,1)$*: for* $x \geq 0$,

$$\boldsymbol{u}^{+}(x) \propto \begin{pmatrix} \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}(1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}} & \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}(1-e^{-x})^{\alpha\rho}(1+e^{-x})^{\alpha\hat{\rho}-1} \\ \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}(1-e^{-x})^{\alpha\hat{\rho}}(1+e^{-x})^{\alpha\rho-1} & \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}(1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho} \end{pmatrix}$$

and

$$\boldsymbol{u}^{-}(x) \propto \begin{pmatrix} \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})}(e^{x}-1)^{\alpha\hat{\rho}-1}(e^{x}+1)^{\alpha\rho} & \frac{\sin(\alpha\pi\hat{\rho})\Gamma(1-\alpha\rho)}{\sin(\alpha\pi\rho)\Gamma(\alpha\hat{\rho})}(e^{x}-1)^{\alpha\hat{\rho}}(e^{x}+1)^{\alpha\rho-1} \\ \frac{\sin(\alpha\pi\rho)\Gamma(1-\alpha\hat{\rho})}{\sin(\alpha\pi\hat{\rho})\Gamma(\alpha\rho)}(e^{x}-1)^{\alpha\rho}(e^{x}+1)^{\alpha\hat{\rho}-1} & \frac{\Gamma(1-\alpha\hat{\rho})}{\Gamma(\alpha\rho)}(e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat{\rho}} \end{pmatrix}.$$

While for $\alpha = 1$ *and symmetric: for* $x \geq 0$,

$$\boldsymbol{u}^{+}(x) = \boldsymbol{u}^{-}(x) \propto \begin{pmatrix} (1-e^{-x})^{-1/2}(1+e^{-x})^{1/2} & (1-e^{-x})^{1/2}(1+e^{-x})^{-1/2} \\ (1-e^{-x})^{1/2}(1+e^{-x})^{-1/2} & (1-e^{-x})^{-1/2}(1+e^{-x})^{1/2} \end{pmatrix}.$$

The constants are determined by requiring that the matrices $\int_{[0,\infty)} dx\, \boldsymbol{u}^{+/-}(x)$ *be stochastic.*

Proposition

$\alpha \in (1,2)$: *for* $x \geq 0$,

$$\boldsymbol{u}^{+}(x)$$
$$\propto \frac{\alpha-1}{2}\begin{pmatrix}(1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}} & (1-e^{-x})^{\alpha\rho}(1+e^{-x})^{\alpha\hat{\rho}-1}\\(1-e^{-x})^{\alpha\hat{\rho}}(1+e^{-x})^{\alpha\rho-1} & (1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho}\end{pmatrix}$$
$$-\frac{(\alpha-1)^2}{2(\lambda+\alpha-1)}\begin{pmatrix}(1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}-1} & (1-e^{-x})^{\alpha\rho-1}(1+e^{-x})^{\alpha\hat{\rho}-1}\\(1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho-1} & (1-e^{-x})^{\alpha\hat{\rho}-1}(1+e^{-x})^{\alpha\rho-1}\end{pmatrix}$$

and

$$\boldsymbol{u}^{-}(x)$$
$$\propto \frac{\alpha-1}{2}\begin{pmatrix}(e^{x}-1)^{\alpha\hat{\rho}-1}(e^{x}+1)^{\alpha\rho} & \frac{\sin(\alpha\pi\hat{\rho})}{\sin(\alpha\pi\rho)}(e^{x}-1)^{\alpha\rho}(e^{x}+1)^{\alpha\rho-1}\\\frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat{\rho})}(e^{x}-1)^{\alpha\rho}(e^{x}+1)^{\alpha\hat{\rho}-1} & (e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat{\rho}}\end{pmatrix}$$
$$-\frac{(\alpha-1)^2}{2(\lambda+\alpha-1)}\begin{pmatrix}(e^{x}-1)^{\alpha\hat{\rho}-1}(e^{x}+1)^{\alpha\rho-1} & \frac{\sin(\alpha\pi\hat{\rho})}{\sin(\alpha\pi\rho)}(e^{x}-1)^{\alpha\hat{\rho}-1}(e^{x}+1)^{\alpha\rho-1}\\\frac{\sin(\alpha\pi\rho)}{\sin(\alpha\pi\hat{\rho})}(e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat{\rho}-1} & (e^{x}-1)^{\alpha\rho-1}(e^{x}+1)^{\alpha\hat{\rho}-1}\end{pmatrix}$$

Lemma (Two key identities)

Recall $\underline{G}_q = \sup\{s < \mathbf{e}_q : \xi_s = \underline{\xi}_s\}$, *and* $\tau_0^- = \inf\{t > 0 : \xi_t < 0\}$.

$$\mathbb{E}_{j,x}\left(f(J_{\underline{G}_q})\right) = \int_{E\times[0,\infty)\times[0,\infty)} U_j^-(d\varphi, dr, dz)e^{-qr}f(\varphi)\left[q\underline{\ell}(\varphi) + \underline{N}_\varphi(1-\exp\{-q\zeta\})\right]$$

$$\mathbb{E}_{j,x}\left(f(J_{\underline{G}_q})1_{\{\mathbf{e}_q<\tau_0^-\}}\right) = \int_{E\times[0,\infty)\times[0,x)} U_j^-(d\varphi, dr, dz)e^{-qr}f(\varphi)\left[q\underline{\ell}(\varphi) + \underline{N}_\varphi(1-\exp\{-q\zeta\})\right]$$

Letting $q \to 0$, *for all* $j \in E$ *and* x

$$1 = \int_{E\times[0,\infty)\times[0,\infty)} U_j^-(d\varphi, dr, dz)\underline{N}_\varphi(\zeta = \infty)$$

$$\mathbb{E}_{j,x}\left(\tau_0^- = \infty, f(J_{\underline{G}_\infty})\right) = \int_{E\times[0,\infty)\times[0,x)} U_j^-(d\varphi, dr, dz)f(\varphi)\underline{N}_\varphi(\zeta = \infty)$$

Let $X_{\underline{m}}$ be the point of closest reach of the origin. We observe that

$$U^-_{1,1}(0,x)\underline{N}_1(\zeta=\infty) = \mathbb{P}_{1,x}(\tau_0^- = \infty; J_{\underline{G}_\infty} = 1) = \mathbb{P}_{e^x}(X_{\underline{m}} > 1),$$

with

$$U^-_{1,1}(0,x) = \int_{E\times[0,\infty)\times[0,x)} U^-_1(d\varphi, dr, dz) 1_{\{\varphi=1\}}$$

Lemma (Kyrpianou, Pardo and Watson (2014))

Assume $\alpha \in (0,1)$. *Let* $\tau^{(-1,1)} := \inf\{t \geq 0 : |X_t| < 1\}$. *We have that, for* $x > 1$,

$$\mathbb{P}_x(\tau^{(-1,1)} = \infty) = \Phi(x),$$

$$\Phi(x) = \frac{\Gamma(1-\alpha\rho)}{\Gamma(\alpha\hat{\rho})\Gamma(1-\alpha)} \int_0^{(x-1)/(x+1)} t^{\alpha\hat{\rho}-1}(1-t)^{-\alpha}\, dt.$$

Furthermore, the invariance under translation and the scaling property implies

$$\mathbb{P}_x(\tau^{(-u,v)} = \infty) = \Phi\left(\frac{2x + u - v}{u + v}\right)$$

Let $\underline{m}^+$ and $\underline{m}^-$ be the times when X is at the closest point to the origin on the positive and negative side of the origin, respectively. Thus

$$\mathbb{P}_x(|X_{\underline{m}^-}| > u; X_{\underline{m}^+} > v) = \mathbb{P}_x(\tau^{(-u,v)} = \infty) = \Phi\left(\frac{2x + u - v}{u + v}\right),$$

The point of closest reach of the origin $X_{\underline{m}}$ has a law

$$\frac{\mathbb{P}_x(X_{\underline{m}} \in dz)}{dz} = -\frac{\partial}{\partial v}\mathbb{P}(|X_{\underline{m}^-}| > z; X_{\underline{m}^+} > v)|_{v=z}.$$

From there we easily determine the value of

$$U^-_{1,1}(x)\underline{N}_1(\zeta = \infty) = \mathbb{P}_{x,1}(\tau_0^- = \infty; J_{\underline{G}_\infty} = 1) = \mathbb{P}_{e^x}(X_{\underline{m}} > 1)$$

Thank you for your attention

Enjoy the 8th International Conference on Lévy Processes