Exit times for random walks with non-identically distributed increments.

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(based on joint works with Denis Denisov and Alexander Sakhanenko)

One-dimensional random walk with i.i.d. increments.

Let $\{S_n\}_{n\geq 1}$ be a random walk,

$$S_0 = 0, \quad S_n = X_1 + \cdots + X_n,\ n \geq 1,$$

where $\{X_n\}_{n\geq 1}$ are *i.i.d.* random variables.

Let τ_x be the exit time from the positive half-line,

$$\tau_x = \inf\{n \geq 1 : x + S_n \leq 0\},$$

where $x \geq 0$.

Wiener-Hopf factorisation.

For $x = 0$, using *time-reversibility* of random walk and dual stopping time one obtains

$$1 - \mathbf{E}[u^{\tau_0}] = \exp\left\{-\sum_{n=1}^{\infty} \frac{u^n}{n}\mathbf{P}(S_n \le 0)\right\}$$

If $\mathbf{P}(S_n \le 0) \to \rho$ then, applying Tauberian theorem, one can derive asymptotics for $\mathbf{P}(\tau_0 > n)$.

Assume that $\mathbf{E}[X_1] = 0$ and $\mathbf{E}[X_1^2] = 1$. Then $\rho = 1/2$ and

$$\mathbf{P}(\tau_0 > n) \sim \sqrt{\frac{2}{\pi}} \frac{\mathbf{E}[-S_{\tau_0}]}{\sqrt{n}}, \quad n \to \infty.$$

For $x > 0$ one has

$$\mathbf{P}(\tau_x > n) \sim \sqrt{\frac{2}{\pi}} V(x) \frac{1}{\sqrt{n}},$$

where $V(x) = \mathbf{E}[-S_{\tau_x}]$. $V(x)$ can also be described as the renewal function of the decreasing ladder height process.

Conditioned random walk.

Using Wiener-Hopf techniques one can also find the following weak limit

$$\lim_{n\to\infty} \mathbf{P}(S_n > v\sqrt{n} \mid \tau_x > n), \quad v \geq 0.$$

In fact, one can show that the linearly interpolated random walk

$$s(t) := S_n + (t-n)X_{n+1}, \quad t \in [n, n+1],$$

after rescaling $s_n(t) := s(nt)/\sqrt{n}$ converges to the Brownian meander, see Iglehart, 1974 and Bolthausen, 1976.

Summary of the Wiener-Hopf factorisation for random walks.

- Powerful analytical method that allows to analyse exit times from half-lines and related quantities.
  - Transforms of distributions of exit times, overshoots. They can be inverted explicitly in some cases or numerically.
  - Exact asymptotics and asymptotic expansions for exit times .
  - Functional limit theorems for conditioned processes.
- This method is specific to one-dimensional homogeneous random walks. It is not of much help for the analysis of
  - exit times for random walks in higher dimensions,
  - exit times for random walks with non-identically distributed increments,
  - exit times for for Markov chains;

Invariance principles for exit times - version 1.0: large starting points.

Now we will consider (for illustration purposes) the above one-dimensional random walk and discuss how to deal with exit times without the Wiener-Hopf method.

Consider the growing starting point $x\sqrt{n}$. Then, we can make use of the Donsker-Prokhorov invariance principle:

$$s_n(t) := \frac{s(nt)}{\sqrt{n}} \to W_t, \quad t \in [0,1], n \to \infty.$$

This gives immediately

$$\mathbf{P}(\tau_{x\sqrt{n}} > n) = \mathbf{P}(\min_{k \le n} S_k > -x\sqrt{n}) \to \mathbf{P}(\min_{0 \le s \le 1} W_s > -x).$$

Invariance principles for exit times - version 1.0: repulsion

But we want to start from a fixed point x. In this case invariance principle is too inaccurate, so let's wait until random walk $x + S_n$ is far away from 0. Let

$$\nu_n := \min\{k \ge 1 : x + S_k \ge n^{1/2-\varepsilon}\}.$$

Stopping time ν_n is sufficiently small, so

$$\mathbf{P}(\nu_n > n^{1-\varepsilon}, \tau_x > n^{1-\varepsilon}) \le Ce^{-cn^{\varepsilon}}.$$

Then

$$\begin{aligned} \mathbf{P}(\tau_x > n) &= \mathbf{P}(\tau_x > n, \nu_n \le n^{1-\varepsilon}) + O(e^{-cn^{\varepsilon}}) \\ &\approx \mathbf{E}[\mathbf{P}(\widetilde{\tau}_{S_{\nu_n}} > n - n^{1-\varepsilon}), \tau_x > \nu_n] + O(e^{-cn^{\varepsilon}}). \end{aligned}$$

Now, S_{ν_n} is of order $n^{1/2-\varepsilon}$, which is sufficiently large but is still much smaller than $n^{1/2}$.

Invariance principles for exit times - version 1.0: KMT-coupling

Hence, functional limit theorem is not accurate enough and we need to apply more accurate KMT coupling.

$$\mathbf{P}\left(\sup_{u\le n}|s(u)-W_u|\ge n^{1/2-\gamma}\right)\le Cn^{-\widetilde{\varepsilon}}.$$

Since S_{ν_n} is of order $n^{1/2-\varepsilon}$ we obtain

$$\mathbf{P}(\widetilde{\tau}_{S_{\nu_n}}>n-n^{1-\varepsilon})=(1+o(1))\mathbf{P}(\tau^{bm}_{S_{\nu_n}}>n-n^{1-\varepsilon}).$$

Now recall, that for the standard Brownian motion and $x=o(\sqrt{n})$,

$$\mathbf{P}(\tau^{bm}_x>n)\sim\sqrt{\frac{2}{\pi}}\frac{x}{\sqrt{n}},\quad n\to\infty$$

Therefore,

$$\mathbf{P}(\widetilde{\tau}_{S_{\nu_n}}>n-n^{1-\varepsilon})\sim\sqrt{\frac{2}{\pi}}\frac{S_{\nu_n}}{\sqrt{n}},\quad n\to\infty.$$

Invariance principles for exit times - version 1.0: fixed starting point

Then,

$$\mathbf{P}(\tau_x > n) \sim \mathbf{E}[\mathbf{P}(\widetilde{\tau}_{S_{\nu_n}} > n - n^{1-\varepsilon}), \tau_x > \nu_n]$$
$$\sim \sqrt{\frac{2}{\pi}} \frac{\mathbf{E}[S_{\nu_n}, \tau_x > \nu_n]}{\sqrt{n}}.$$

We are done once we show that

$$\mathbf{E}[S_{\nu_n}, \tau_x > \nu_n] \to V(x).$$

As $S_n I(\tau_x > n)$ is a submartingale it is sufficient to show that

$$\sup_n \mathbf{E}[S_{\nu_n}, \tau_x > \nu_n] < \infty.$$

This can be done via recursive estimates.

Invariance principles for exit times - version 1.0: key steps

1. We found asymptotics for exit times in the Brownian motion.
2. For the random walk we showed repulsion from boundaries.
3. We constructed function $V(x) = -\mathbf{E}[S_{\tau_x}]$
4. We used repulsion from boundaries to replace the random walks with the Brownian motion.
5. KMT coupling was used to control the error of replacement of the random walk with the Brownian motion.

These steps allowed us to transform the results for the exactly solvable exit times for the standard Brownian motion to the approximations for the exit times of random walks.

Invariance principles for exit times - version 1.0: multidimensional models

- Random walks in Weyl chambers, Denisov and Wachtel, 2010.
- Random walks in cones, Denisov and Wachtel, 2015.
- Integrated random walks, Denisov and Wachtel, 2015.
- Conditional limit theorems for products of random matrices, Grama, Le Page and Peigne, 2014

The main difficulty: In order to apply the above methodology one needs to prove functional limit theorem with rate of convergence (construct a coupling).

Invariance principles for exit times - version 2.0: random walks with non-i.i.d. increments

Now we consider

$$S_n = X_1 + \cdots + X_n,$$

where X_n are independent but not identically distributed. Let $g = (g_n)_{n\ge1}$ be a real-valued sequence and let

$$\tau_x := \min\{n \ge 1 : x + S_n \le 0\},$$

$$T_g := \min\{n \ge 1 : S_n \le -g_n\}.$$

We are interested in finding asymptotics for

$$\mathbf{P}(T_g > n), \quad n \to \infty.$$

Invariance principles for exit times - version 2.0: main assumptions

We assume that X_k are centred

$$\mathbf{E}[X_k] = 0, \quad k \geq 1$$

and have finite variances

$$\sigma_k^2 := \mathbf{E}[X_k^2] \in (0, \infty).$$

Let

$$B_n^2 := \sum_{k=1}^{n} \sigma_k^2, \quad n \geq 1,$$

and assume that *the Lindeberg condition* holds, i.e.

(1) $$L_n^2(\varepsilon) := \frac{1}{B_n^2} \sum_{k=1}^{n} \mathbf{E}[X_k^2; |X_k| > \varepsilon B_n] \to 0, \quad \text{for every } \varepsilon > 0.$$

Invariance principles for exit times - version 2.0: functional CLT

Interpolate the random walk as follows,

$$s(t) = S_k + X_{k+1} \frac{t - B_k^2}{\sigma_{k+1}^2}, \quad t \in [B_k^2, B_{k+1}^2], k \geq 1$$

and let

$$s_n(t) := \frac{s(tB_n^2)}{B_n^2}.$$

Then, under the Lindeberg condition (1),

$$s_n(t) \stackrel{d}{\to} W_t, \quad n \to \infty,$$

in $C[0,1]$ endowed with the supremum norm.

Invariance principles for exit times - version 2.0: asymptotics far away from the boundary

For $x = uB_n$, we can apply the functional limit theorem as above

$$\mathbf{P}(\tau_x > n) = \mathbf{P}(x + \min_{k\le n} S_k > 0) = \mathbf{P}(u + \min_{k\le n} S_k/B_n > 0)$$
$$\to \mathbf{P}(u + \min_{t\le 1} W_t > 0) = \mathbf{P}(\tau_u^{bm} > 1) = \mathbf{P}(\tau_x^{bm} > B_n).$$

We can slightly improve by considering $x = u_n B_n$, where $u_n \to 0$ sufficiently slowly:

$$\mathbf{P}(\tau_{x_n} > n) \sim \mathbf{P}(\tau_{x_n}^{bm} > B_n) \sim \sqrt{\frac{2}{\pi}}\frac{x_n}{B_n}.$$

Invariance principles for exit times - version 2.0: main result

Theorem 1.

Let $\mathbf{P}(T_g > n) > 0, n \geq 1$ and

$$g_n = o(B_n).$$

If the Lindeberg condition (1) holds then

$$\mathbf{P}(T_g > n) \sim \sqrt{\frac{2}{\pi}} \frac{U_g(B_n^2)}{B_n},$$

where U_g is a positive, *slowly varying function*.

Furthermore $s_n(t)$ conditioned on $\{T_g > n\}$ converges weakly in $C[0,1]$ towards the Brownian meander. In particular,

$$\mathbf{P}\left(\frac{S_n}{B_n} > v \mid T_g > n\right) \to e^{-v^2/2}, \quad v \geq 0.$$

Weighted random walks

Aurzada and Baumgarten, 2011 considered weighted random walks

$$S(n) = \sum_{k=1}^{n} \sigma(k)\xi_k,$$

where ξ_k are i.i.d. with mean $\mathbf{E}[\xi_k] = 0$ and $Var(\xi_k) < \infty$. Under the assumption $\mathbf{E}[e^{\lambda|\xi_1|}] < \infty$ for some $\lambda > 0$ they showed that if $\sigma(n) = n^{p+o(1)}$ then

$$\mathbf{P}(\tau_0 > n) \sim n^{-p+1/2+o(1)}, \quad n \to \infty.$$

To obtain this result they used the KMT coupling.

Our Theorem 1 states that the same asymptotic behaviour takes places for all weighted random walks with fine variances and for all boundaries $g_n = o(B_n)$.

Invariance principles for exit times - version 2.0: key changes

Lindeberg condition implies weak convergence in C[0,1] of $s_n(t)$ to W_t.

Hence, we can use results of Strassen, 1965 and Skorokhod, 1977 to construct the following coupling: one can construct S_n and Brownian motion $W_n(t)$ on a common probability space so that

$$\mathbf{P}\Big(\max_{0\le t\le B_n^2} |s(t) - W_n(t)| > \pi_n B_n\Big) \le \pi_n,$$

where $\pi_n \to 0$.

Instead ν_n in version 1.0 we use

$$\nu(h) := \inf\{k \ge 1 : S_k > g_k + h\} \text{ and } \nu_m := \min(\nu(B_m), m).$$

For every sequence π_n we can find $m(n)$ such that random walk and Brownian motion are suficiently close after reaching the level $B_{m(n)}$. This means that we may apply coupling at time $\nu_{m(n)}$ if $S_{\nu_{m(n)}} \ge B_{n(m)}$.

Asymptotic behaviour of U_g: question

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Can we strengthen the statement of Theorem 1 and say that $U_g(B_n^2) \to C \in (0,\infty)$?

The answer is NO.

Novikov, 1983 has shown that for random walks with i.i.d. increments and increasing boundaries g_n one has

$$\mathbf{P}(T_g > n) \sim \frac{C_g}{\sqrt{n}} \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{g_n}{n^{3/2}} < \infty.$$

Convergence of U_g.

Theorem 2. Let the assumptions of Theorem 1 hold and in addition $\bar{g} := \sup_n g_n < \infty$. Then the expectation $E[-S_{T_g}]$ is well defined and

$$\lim_{x\to\infty} U_g(x) = \mathbf{E}[-S_{T_g}] \in (0,\infty].$$

In particular, if $\mathbf{E}[-S_{T_g}]$ is finite we have the following *exact* asymptotics

$$\mathbf{P}(T_g > n) \sim \frac{-\mathbf{E}[S_{T_g}]}{B_n}, \quad n \to \infty.$$

Necessary condition for $E[-S_{\tau_x}] < \infty$.

Assume now that $g_n \equiv -x$. Is $\lim_{n\to\infty} U_g(n) \in (0,\infty)$?

The answer is again NO and the following necessary condition should hold.

If $E[-S_{\tau_x}]$ is finite then

$$\sum_{n=1} \frac{1}{B_n}\mathbf{E}[-X_n; -X_n > \varepsilon B_n] < \infty \text{ for each } \varepsilon > 0.$$

Necessary condition for $E[-S_{\tau_x}] < \infty$: example.

Let X_n be a symmetric random variable with four values:

$$\mathbf{P}(X_n = \pm\sqrt{n}) = \frac{p_n}{2}, \quad \mathbf{P}(X_n = \pm a_n) = \frac{1-p_n}{2},$$

where

$$p_n := \frac{1}{n\log(2+n)} \quad \text{and} \quad a_n := \sqrt{\frac{1-np_n}{1-p_n}}.$$

Clearly, $\mathbf{E}X_n = 0$ and $\mathbf{E}X_n^2 = 1$. Therefore, $B_n = \sqrt{n}$ for this sequence of random variables.

This sequence satisfies the Lindeberg condition.

For $\varepsilon = 1/2$ we have

$$\sum_{k=2}^{\infty} \frac{1}{B_k} \mathbf{E}[-X_k; -X_k > B_k/2] = \sum_{k=2}^{\infty} \frac{1}{\sqrt{k}} \sqrt{k} p_k = \sum_{k=2}^{\infty} \frac{1}{k \log(2+k)} = \infty.$$

This implies that $\mathbf{E}[-S_{T_g}] = \infty$ and, consequently,

$$\sqrt{n} \mathbf{P}(\tau_0 > n) \to \infty.$$

Sufficient conditions for $E[-S_{\tau_x}] < \infty$.

Theorem 3. Assume that the *uniform* Lindeberg condition holds. Assume also that

$$\sum_{k=1}^{\infty} \frac{\log^{1+\gamma} B_k^2}{B_k} \mathbf{E}\left[-X_{k+1}; -X_{k+1} > \frac{B_k}{\log^{2+2\gamma} B_k^2}\right] < \infty$$

and

$$g_n > -a \frac{B_n}{\log^{2+2\gamma} B_n}$$

for some $\gamma > 1$ and some $a > 1$. Then,

$$\lim_{n \to \infty} U_g(n) = \mathbf{E}[-S_{T_g}] < \infty.$$

<u>Possible generalisations.</u>

- Asymptotically stable random walks with non-identically distributed increments.
- One-dimensional Markov chains.
- One-dimensional martingales, stationary sequences.
- Markov chains in cones.