Absorption problem for multidimensional random walks

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Absorption problem for random walks

Random walk in $\mathbb{R}^1$

Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with a symmetric density f:

$$f(-t) = f(t).$$

The partial sums $S_k = X_1 + \ldots + X_k$ form a random walk.

Theorem (Sparre Andersen, 1949)

The probability that such random walk stays positive by the time n is given by

$$\mathbb{P}[S_1 > 0, \ldots, S_n > 0] = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n-1)!!}{2^n n!}.$$

- This formula is distribution-free!
- This is a combinatorial result.

Random walk in $\mathbb{R}^d$

Let the increments $X_1, \dots, X_n$ be i.i.d. random vectors in $\mathbb{R}^d$ with a centrally symmetric density $f(t) = f(-t)$.

Problem

Compute the **absorption probability**

$$\mathbb{P}[0 \in \operatorname{conv}\{S_1, S_2, \dots, S_n\}].$$

Note that for $d = 1$,

$$\mathbb{P}[0 \notin \operatorname{conv}\{S_1, S_2, \dots, S_n\}] = 2\mathbb{P}[S_1 > 0, \dots, S_n > 0].$$

References

Transition from non-absorption to absorption in high dimension d as the number of steps n increases:
Eldan (2014), Tikhomirov and Youssef (2014, 2015).

Who cares?

- Generalization of the persistence problem to $\mathbb{R}^d$.
- Connection with integral geometry via the conic Crofton formula.
- Wendel's formula (1962) for absorption probability for i.i.d. random vectors with centrally symmetric density.

Theorem (V. and D. Zaporozhets, 2015) for $d = 2$

For random walks with centr. symmetric density of increments,

$$\mathbb{P}[0 \notin \mathrm{conv}\{S_1, S_2, \dots, S_n\}] = \sum_{k=1}^{n} \frac{(2n-2k-1)!!}{k(2n-2k)!!}.$$

Conjecture (V. and D. Zaporozhets, 2015)

The absorption probability is distribution free in $\mathbb{R}^d$.

Our method does not work for $d \geq 3$ but it allows to find the asymptotics of absorption probabilities for $d = 2$.

Main result

Theorem (Kabluchko, V., and Zaporozhets 2015+)

For random walks with centr. symmetric density of increments,

$$\mathbb{P}[0 \in \mathrm{conv}(S_1, \dots, S_n)] = \frac{2}{2^n n!}(b_n(d+1) + b_n(d+3) + \dots),$$

where $b_n(k)$ are the coefficients of the polynomial

$$(x+1)(x+3)\dots(x+2n-1) = \sum_{k=0}^{\infty} b_n(k) x^k.$$

Connections

- Combinatorics: $b_n(k)$ are known as the b-analogues of the Stirling numbers of the first kind.
- Geometry: $b_n(k)$ are the **conic intrinsic volumes** of Weyl chambers of type B_n.

Asymptotics

This explicit formula is very tractable and allows to compute the asymptotics for both fixed and increasing dimensions.

Theorem (fixed dimension d)

Under our assumptions, as $n \to \infty$,

$$\mathbb{P}[0 \notin \mathrm{conv}\{S_1, \dots, S_n\}] \sim \frac{2(\log n)^{d-1}}{2^{d-1}(d-1)!\sqrt{\pi n}}.$$

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Critical number of steps in high dimension

Consider a **high** dimension d. It is clear that

- If n is "small", then non-absorption has high probability.
- If n is "large", then absorption has high probability.

There is a critical value $n = n(d)$ for which the absorption probability is $\approx 1/2$.

Previous results

- Eldan (2014): $e^{c_1 d/\log d} \leq \text{critical } n \leq e^{c_2 d \log d}$.
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Theorem (CLT in high dimensions, K.–V.–Z. (2015+))

Let the dimension $d = d(n)$ be such that for some $a \in \mathbb{R}$,

$$d(n) = \frac{1}{2}\log n + a\sqrt{\frac{1}{2}\log n} + o(\sqrt{\log n}),$$

as $n \to \infty$. Then under our assumptions,

$$\lim_{n\to\infty} \mathbb{P}[0 \notin \operatorname{conv}\{S_1, \ldots, S_n\}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \mathrm{e}^{-t^2/2} \mathrm{d}t.$$

Hence the critical number of steps is $n \approx \mathrm{e}^{2d}$.

Proof

Symmetric exchangeability of the increments

For every permutation σ on $\{1, \ldots, n\}$ and for every choice of the signs we have

$$(X_1, \ldots, X_n) \overset{d}{=} (\pm X_{\sigma(1)}, \ldots, \pm X_{\sigma(n)}).$$

Symmetry group B_n

The group B_n has $2^n n!$ elements which act on $\mathbb{R}^n$ by

- permuting the coordinates
- changing the sings of any number of coordinates.

Weyl chambers of type B_n

The standard **Weyl chamber** of type B_n is the cone

$$W_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \colon 0 < x_1 < \ldots < x_n\}.$$

Any of $2^n n!$ cones $gW_n, g \in B_n$, is called a Weyl chamber.

Probabilistic problem $\Leftrightarrow$ Geometric problem(s)

Absorption occurs iff for some non-trivial $\alpha_1, \ldots, \alpha_n \geq 0$:

$$\alpha_1 X_1 + \alpha_2 (X_1 + X_2) + \ldots + \alpha_n (X_1 + \ldots + X_n) = 0.$$

Rearrange the terms:

$$X_n \underbrace{\alpha_n}_{\beta_1} + X_{n-1} \underbrace{(\alpha_n + \alpha_{n-1})}_{\beta_2} + \ldots + X_1 \underbrace{(\alpha_1 + \ldots + \alpha_n)}_{\leq \beta_n} = 0.$$

Thus $(\beta_1, \ldots, \beta_n)$ belongs to the standard Weyl chamber $\overline{W_n}$.

On the other hand, $\beta_n X_1 + \beta_{n-1} X_2 + \ldots + \beta_1 X_n = 0$ means

$$(\beta_1, \ldots, \beta_n) \in Ker\ A,$$

where A is the $d \times n$ matrix with columns $X_n, \ldots, X_1$.

Random linear subspace

$$\mathbb{P}[0 \in \mathrm{conv}(S_1, \ldots, S_n)] = \mathbb{P}[\mathit{Ker}\ A \text{ intersects } \overline{W_n}\,]$$

Ker A is a *random linear subspace* of $\mathbb{R}^n$ of codimension d a.s.

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Conic Crofton formula

Let L be a uniformly distributed random linear subspace of $\mathbb{R}^n$ of codimension d. Then for every convex cone C,

$$\mathbb{P}[L \text{ intersects } C] = 2(\nu_{d+1}(C) + \nu_{d+3}(C) + \ldots),$$

where $\nu_k(C)$ are conic intrinsic volumes of C.

If S_k is a standard Gaussian random walk in $\mathbb{R}^d$, then *Ker A* $\overset{d}{=} L$.

Theorem (Klivans–Swartz (2011), K.–V.–Z. (2015+))

The conic intrinsic volumes $\nu_k(W_n)$ are the coefficients of the polynomial $\frac{(x+1)(x+3)\ldots(x+2n-1)}{2^n n!}$. That is $\nu_k(W_n) = b_n(k)$.

Solving the geometric problem

The B_n-invariance (symmetric exchangeability) implies that *Ker A* intersects every Weyl chamber with the same probability.

Basic lemma

Let $N_{n,d}$ be the number of Weyl chambers of type B_n intersected by *Ker A*. Then

$$\mathbb{P}[0 \in \operatorname{conv}(S_1, S_2, \ldots, S_n)] = \frac{\mathbb{E} N_{n,d}}{2^n n!}.$$

Solution by the theory of hyperplane arrangements

- The walls of the Weyl chambers are formed by the hyperplanes $x_i = 0$ and $x_i = \pm x_j$, $1 \le i < j \le n$.
- Every subspace of codimension d in "general position" intersects the **same** number of chambers! So $N_{n,d} = const$.

Invariance for other symmetry groups

Symmetry group A_{n-1}

The group A_{n-1} acts on $\mathbb{R}^n$ by permuting the coordinates in an arbitrary way. The number of elements is $n!$. The hyperplane $x_1 + \ldots + x_n = 0$ stays invariant.

Random walk bridge

- Let $X_1, \ldots, X_n$ be i.i.d. random vectors in $\mathbb{R}^d$ with arbitrary density (no symmetry assumption!).
- Consider the partial sums $S_k = X_1 + \ldots + X_k$.
- We are interested in the **random walk bridge**: $S_1, S_2, \ldots, S_{n-1}$ conditioned on $S_n = 0$.

Absorption probability for random walk bridges

Theorem (Kabluchko, V., and Zaporozhets 2015+)

$$\mathbb{P}[0 \in \mathrm{conv}(S_1, \dots, S_{n-1}) | S_n = 0] = \frac{2}{n!}\left(\begin{bmatrix} n \\ d+2 \end{bmatrix} + \begin{bmatrix} n \\ d+4 \end{bmatrix} + \dots \right)$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ are the **Stirling numbers** of the first kind defined by

$$x(x+1)\dots(x+n-1) = \sum_{k=1}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} x^k.$$

- $\begin{bmatrix} n \\ k \end{bmatrix}$ is the number of n-permutations with exactly k cycles.
- $\begin{bmatrix} n \\ k \end{bmatrix}$ is the probability that there are exactly k records in an i.i.d. sample of size n from a continuous distribution.

Special case $d = 1$ (Sparre Andersen, 1953)

$$\mathbb{P}[S_1 > 0, \dots, S_{n-1} > 0 | S_n = 0] = 1/n.$$

D_n-invariance: random walks flipping the last jump

Symmetry group D_n

The group D_n acts on $\mathbb{R}^n$ by permuting the coordinates in an arbitrary way and by changing the sign of an **even** number of coordinates. The number of elements is $2^{n-1}n!$.

Probabilistic problem corresponding to D_n

- Let $X_1, \ldots, X_n$ be independent identically distributed random vectors in $\mathbb{R}^d$ with centrally symmetric density.
- Consider the partial sums $S_k = X_1 + \ldots + X_k$ and

$$S_n^* = X_1 + \ldots + X_{n-1} - X_n.$$

- We can compute the probability that $\mathrm{conv}(S_1, \ldots, S_n, S_n^*)$ contains 0.

Invariance under products of symmetry groups

Type $B_{n_1} \times \ldots \times B_{n_r}$

Take r independent, symmetric random walks with densities in $\mathbb{R}^d$ with $n_1, \ldots, n_r$ steps. We can compute the probability that the joint convex hull of these random walks absorbs the origin.

Example: $B_1^r = (\mathbb{Z}/2\mathbb{Z})^r$ (Wendel, 1962)

Let $X_1, \ldots, X_r$ be i.i.d. random vectors in $\mathbb{R}^d$ with a centrally symmetric density. Then,

$$\mathbb{P}[0 \notin \operatorname{conv}\{Z_1, \ldots, Z_r\}] = \frac{1}{2^{r-1}} \sum_{k=0}^{d-1} \binom{r-1}{k}.$$

Removing the density assumption

All the stated results are valid for partial sums S_k of increments whose joint distribution has density and is invariant under the action of the corresponding group A_{n-1}, B_n, D_n or $B_{n_1} \times \ldots \times B_{n_r}$.

Example

- The existence of density is essential!
- For a random walk with

$$\mathbb{P}[X_1 = 1] = \mathbb{P}[X_1 = -1] = 1/4, \mathbb{P}[X_1 = 0] = 1/2,$$

the non-absorption probability equals $\frac{1}{2}$ of the non-absorption probability for a walk with symmetric density.

Theorem (Kabluchko, V., Zaporozhets 2015+)

The absorption probabilities are minimal if the increments have a joint density (and satisfy the respective assumptions).

THANK YOU!