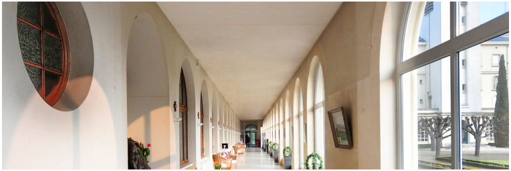

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New results related to complete monotonicity and Mellin transform, with applications to infinite divisibility

Plan

1. Contribution to Complete monotonicity and alternation of functions
2. Contribution to properties of Mellin transform
3. A Normal Limit Theorem for size biasing

Operators of interest

Let the operators

$$\begin{aligned} \Delta_c f(x) &:= f(x+c) - f(x), & \Delta := \Delta_1, \\ \theta_c f(x) &:= f(c) - f(0) + f(x) - f(x+c), & \theta := \theta_1, \end{aligned}$$

Their iterates are given by $\Delta_c^0 f = \theta_c^0 f = f$ and for every $n \in \mathbb{N}$,

$$\Delta_c^n f = \Delta_c(\Delta_c^{n-1} f), \quad \theta_c^n f = (-1)^n \big(\Delta_c^n f - \Delta_c^n f(0)\big),$$

so that for every $n \in \mathbb{N}_0$,

$$\Delta_c^n f(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} f(x+ic) \tag{1}$$

$$\theta_c^n f = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i} \Big(f(x+ic) - f(ic) \Big). \tag{2}$$

Completely monotone and alternating functions

Definition (Berg & Christensen 1984)

Let $D = (0, \infty)$ or $[0, \infty)$ or $\mathbb{N}_0$. A function $f : D \to f(D)$ is called completely monotone on D, we denote $f \in \mathcal{CM}(D)$, if $f(D) \subset [0, \infty)$ (respectively completely alternating, we denote $f \in \mathcal{CA}(D)$, if $f(D) \subset \mathbb{R}$), and for if all finite sets $\{c_1, \cdots, c_n\} \subset D$ and $x \in D$, we have

$$(-1)^n \Delta_{c_1} \cdots \Delta_{c_n} f(x) \geq 0 \quad (\textit{respectively} \ \leq 0).$$

Theorem (Bernstein's characterization of $\mathcal{CM}(0,\infty)$)

$$\begin{aligned} f \in \mathcal{CM}(0,\infty) &\iff (-1)^n f^{(n)}(\lambda) \geq 0, \quad \forall n \geq 0 \textit{ and } \lambda > 0 \\ &\iff f(\lambda) = \int_{[0,\infty)} e^{-\lambda x} \nu(dx). \end{aligned}$$

Theorem

The class of Bernstein functions $\mathcal{BF}$ is given by

$$\begin{aligned} \mathcal{BF} &= \{\phi : (0,\infty) \to (0,\infty),\ \textit{differentiable},\ s.t.\ \phi' \in \mathcal{CM}(0,\infty)\} \\ &= \{\phi(\lambda) = q + d\,\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\,\pi(\mathrm{d}x),\ \lambda \geqslant 0\} \\ &= \mathcal{CA}(0,\infty) \cap \{\phi \geq 0\}. \end{aligned}$$

Note that $f \in \mathcal{CM}(0,\infty) \Longrightarrow -f \in \mathcal{CA}(0,\infty)$

Theorem (Hausdorff's characterization of $\mathcal{CM}(\mathbb{N})$ and $\mathcal{CA}(\mathbb{N})$)

$$\begin{aligned}
(a_k)_{k\geq 0} \in \mathcal{CM}(\mathbb{N}_0) \quad &\iff \quad (-1)^n \Delta^n a(k) \geq 0,\ \forall k \in \mathbb{N}_0,\ n \in \mathbb{N}_0 \\
&\iff \quad a_0 = \nu([0,1]), \quad a_k = \int_{(0,1]} u^k \nu(\mathrm{d}u), \quad k \in \mathbb{N}_0 \\
(a_k)_{k\geq 0} \in \mathcal{CA}(\mathbb{N}_0) \quad &\iff \quad (-1)^n \Delta^n a(k) \leq 0,\ \forall k \in \mathbb{N}_0,\ n \in \mathbb{N} \\
&\iff \quad a_k = q + d\,k + \int_{[0,1)} (1 - u^k)\,\mu(\mathrm{d}u), \quad k \in \mathbb{N}_0.
\end{aligned}$$

Definition : A sequence $a = (a_k)_{k\geq 0}$ is called minimal (and we denote $a \in \mathcal{CM}^*(\mathbb{N}_0)$, resp. $a \in \mathcal{CA}^*(\mathbb{N}_0)$) if the sequence

$$\{a_0 - \epsilon, a_1, \cdots, a_k, \cdots\} \quad (\text{resp. } \{a_0, a_1 - \epsilon, \cdots, a_k - \epsilon, \cdots\})$$

is not in $\mathcal{CM}(\mathbb{N}_0)$ (resp. $\mathcal{CA}(\mathbb{N}_0)$) for any $\epsilon > 0$.

Theorem (Widder 1946 and Athavale-Ranjekar 2002)

A sequence a is minimal IFF $\nu(\{0\}) = 0$ (resp. $\mu(\{0\}) = 0$).

FACT and QUESTION

Fact : *Minimal completely monotone (resp. alternating) functions are interpolated by functions in $\mathcal{CM}$ (resp. $\mathcal{BF}$) !*

Question : *Can we affirm that a function f is $\mathcal{CM}$ (resp. $\mathcal{BF}$) if we know that the sequence $(f(k))_k$ is $\mathcal{CM}$ (respectively $\mathcal{CA}$) ?*

Reformulation : *Could a $\mathcal{CM}$ (resp. $\mathcal{CA}$) and minimal sequence $(a_k)_k$ be interpolated by a regular enough function f, which is not $\mathcal{CM}$ (respectively $\mathcal{BF}$) ?*

This is a kind of converse to Hausdorff's moment characterization problem !

ANSWER for $\mathcal{CM}$

Theorem (Characterization of $\mathcal{CM}$)

A function $\Psi \in \mathcal{CM}(\mathbb{R}_+)$ IFF the two following conditions hold :

(i) Ψ have an holomorphic bounded extension on $Re(z) > 0$;

(ii) the sequence $\big(\Psi(k)\big)_{k\geq 0} \in \mathcal{CM}^*(\mathbb{N}_0)$.

Compare with

Theorem (Mai, Schenk, Scherer 2015)

$\Psi \in \mathcal{CM}(\mathbb{R}_+)$ IFF Ψ is continuous and

$\big(\Psi(xk)\big)_{k\geq 0} \in \mathcal{CM}^*(\mathbb{N}_0), \quad \forall x \in \mathbb{Q} \cap (0,\infty).$

Additional Results on $\mathcal{CM}$

Corollary

$\Psi \in \mathcal{CM}(0,\infty)$ *IFF for some (and hence all) sequence* $\epsilon_n \searrow 0$,

(i) Ψ *has a holomorphic bounded extension on* $Re(z) > \epsilon_n$ *;*

(ii) $\big(\Psi(\epsilon_n + k)\big)_{k\geq 0} \in \mathcal{CM}^\star(\mathbb{R}_+)$.

Corollary

Two functions in $\mathcal{CM}(0,\infty)$ *coincide on* $\mathbb{N}$ *starting from a certain rank IFF they are equal.*

Answer for $\mathcal{BF}$

Theorem (Characterization of $\mathcal{BF}$)

A function $\Phi : [0,\infty) \to [0,\infty)$ belongs to $\mathcal{BF}$ if and only if

(i) it has an holomorphic extension on $Re(z) > 0$, s.t. $|\Phi(c+z) - \Phi(z)| \leq M$ for some c, $M > 0$

(ii) $\big(\Phi(k)\big)_{k\geq 0} \in \mathcal{CA}^\star(0,\infty)$.

Corollary

A function $\Phi : [0,\infty) \longrightarrow [0,\infty)$ belongs to $\mathcal{BF}$ IFF it is continuous and for each $n \in \mathbb{N}$, $\big(\Phi(\frac{k}{n})\big)_{k\geq 0} \in \mathcal{CA}^*(\mathbb{N}_0)$.

Corollary

Two Bernstein functions coincide on $\mathbb{N}_0$ starting from a certain rank if and only if they are equal on $[0,\infty)$.

Tools

Proposition

(a) $\Psi \in \mathcal{CM}(0,\infty)$ *IFF for some (and hence for all)* $c > 0$

$$\lambda \mapsto -\Delta_c\Psi(\lambda) = \Psi(\lambda) - \Psi(\lambda + c) \in \mathcal{CM}(0,\infty).$$

(b) If $-\Delta_c\Psi \in \mathcal{CM}(0,\infty)$, *then* $(-\Delta_{nc})\Psi$ *converges pointwise, loc. uniformly, to* Ψ. *The same holds for the successive derivatives of* $(-\Delta_{nc})\Psi$.

Proposition

(a) $\Phi \in \mathcal{BF}$ *IFF for some (and hence for all)* $c > 0$,

$$\lambda \mapsto \theta_c\Phi(\lambda) = \Phi(c) - \Phi(0) + \Phi(\lambda) - \Phi(\lambda + c) \in \mathcal{BF}_b.$$

(b) If $\theta_c\Phi \in \mathcal{BF}$, *then* $\theta_{nc}\Phi$ *converges pointwise, loc. uniformly, to* Φ. *The same holds for the successive derivatives of* $\theta_{nc}\Phi$.

Tools

Lemma (Karamata's Theorem improved)

Suppose $h, l : [0,\infty) \to [0,\infty)$ *are linked for every* $\lambda \geq 0$ *by*

$$h(\lambda + n) - h(n) \to l(\lambda), \quad \text{as } n \to \infty \text{ and } n \in \mathbb{N}.$$

Then, necessarily $l(\lambda) = \lambda\, l(1)$ *with* $l(1) \geq 0$ *and*

$$h(\lambda + x) - h(x) \to l(\lambda), \quad \text{as } x \to \infty,$$

uniformly in each compact λ*-set in* $[0,\infty)$.

Corollary (to Blaschke's theorem)

Two holomorphic functions on $P = \{Re(z) > 0\}$ *are identical if their difference is bounded and they coincide along a sequence* $z_1, z_2, z_3, \cdots$ *in* P, *s.t.* $\sum(1 - |\frac{z_i - 1}{z_i + 1}|) = +\infty$ *(in particular for* $z_i = i \in \mathbb{N}$*).*

Alternative Tools

Theorem (Webster 1997)

Let $g : [0,\infty) \to [0,\infty)$ *log-concave s.t.* $\lim_{a\to\infty} \frac{g(x+a)}{g(a)} = 1$, $\forall a > 0$. *Let* $a_n = (g'_-(n) + g'_+(n))/2g(n)$ *and* $\gamma_g = \lim_{n\to\infty} \left(\sum_1^n a_j - \log g(n)\right)$.
Then, there exists a unique log-convex solution f *to the iter. equation :*

$$f(x+1) = g(x)f(x), \quad x > 0, \quad \text{and} \quad f(1) = 1,$$

given by $f(x) = \frac{e^{-\gamma_g x}}{g(x)} \prod_{n=1}^{\infty} \frac{g(n)}{g(n+x)} e^{a_n x}, \quad x > 0.$

Theorem (Nörlund 1926, on Gregory-Newton development)

f admits a Gregory-Newton development

$$f(z) = \sum_{k=0}^{\infty} (-1)^k \frac{a_k}{k!} z(z-1)\cdots(z-k+1),$$

IFF f is holomorphic on $Re(z) > \alpha$ *and* $\left|f(z)\right| \le Ae^{B|z|}$, $\alpha, A, B > 0$.

Necessarily $a_k = (-1)^k \Delta^k f(0), \quad k \ge 0.$

The Mellin transform, first properties

From now on : all random variables are nonnegative.

$$\mathcal{M}_X(\lambda) = \mathbb{E}[X^\lambda], \quad \text{for } \lambda \text{ in some domain of definition} \subseteq \mathbb{C},$$

$$\mu_X = \inf\{\lambda \in \mathbb{R},\ \mathbb{E}[X^\lambda] < \infty\} \quad \text{and} \quad \lambda_X = \sup\{\lambda \in \mathbb{R},\ \mathbb{E}[X^\lambda] < \infty\}.$$

Proposition

If $\lambda_X > 0$, then, on $[0, \lambda_X]$, $\mathcal{M}_X$ is log-convex and strictly log-convex if X is non-deterministic.

Corollary

Assume $\lambda_X > 0$. For every $\lambda \in (0, \lambda_X)$, the function $t \mapsto \mathcal{M}_X(\lambda + t)/\mathcal{M}_X(t)$ is nondecreasing on $[0, \lambda_X - \lambda)$. It is further increasing whenever X is non-deterministic.

The Mellin transform, injectivity and size biasing

Lemma (Widder's Theorem improved)

Assume $M_X = M_Y$ on some interval $(\alpha, \beta) \subset \mathbb{R}$, then $X \stackrel{d}{=} Y$.

From now on : $\lambda_X > 0$;

The biased law of order $t \in [0, \lambda_X)$ is denoted $X_{(t)}$:

$$\mathbb{P}(X_{(t)} \in dx) = \frac{x^t}{\mathbb{E}[X^t]} \mathbb{P}(X \in dx), \quad x \geq 0. \tag{3}$$

(P0) $X_{(0)} = X$ and $(cX)_{(t)} = cX_{(t)}, \ c > 0$.

(P1) $\mathbb{E}[X_{(t)}^\lambda] = \frac{\mathbb{E}[X^{t+\lambda}]}{\mathbb{E}[X^t]}, \qquad \mathbb{E}[g(X_{(t)})] = \frac{\mathbb{E}[X^t g(X)]}{\mathbb{E}[X^t]}$.

(P2) $\left(X_{(s)}\right)_{(t)} \stackrel{d}{=} X_{(s+t)} \stackrel{d}{=} \left(X_{(t)}\right)_{(s)}$.

(P3) $(X^s)_{(t)} \stackrel{d}{=} (X_{(st)})^s$.

(P4) $X, \ Y$ independent $\Longrightarrow (XY)_{(t)} \stackrel{d}{=} X_{(t)} Y_{(t)}$ ($X_{(t)}$ and $Y_{(t)}$ independent).

Convergence of sequences and families of Mellin transforms

$\mathbb{T} = \mathbb{N}$ or $[0, \infty)$. A subsequence of $(X_t)_{t \in \mathbb{T}}$, is a sequence $(X_{t(n)})_{n \in \mathbb{N}}$ with function $t : \mathbb{N} \to \mathbb{T}$ s.t. $t(n) \nearrow \infty$.

Definition (Billingsley)

(i) $(X_n)_{n \in \mathbb{N}}$ is tight if $\sup_{n \in \mathbb{N}} \mathbb{P}(|X_n| > x) \to 0$ as $x \to \infty$.
(ii) $(X_n)_{n \in \mathbb{N}}$ is UI if $\sup_{n \in \mathbb{N}} \mathbb{E}[|X_n| \mathbb{1}_{X_n > x}] \to 0$ as $x \to \infty$.

Definition

(i) $(X_t)_{t \in \mathbb{T}}$ is ultimately tight if $\limsup_{t \in \mathbb{T}} \mathbb{P}(X_t > x) \to 0$, as $x \to \infty$.
(ii) $(X_t)_{t \in \mathbb{T}}$ is λ-UI if $\limsup_{t \in \mathbb{T}} \mathbb{E}[X_t^\lambda \mathbb{1}_{X_t > x}] \to 0$, as $x \to \infty$.
(iii) $X_t \xrightarrow{d} X_\infty$, if $X_{t(n)} \xrightarrow{d} X_\infty$ in the usual sense for every $(t(n))_n$.

Convergence of sequences and families of Mellin transforms

Theorem

1) Let $X_\infty \geq 0$. *The following assertions are equivalent, as* $t \to \infty$:

(i) $X_t \xrightarrow{d} X_\infty$ *and* $(X_t)_{t\in\mathbb{T}}$ *is* λ_o*-UI ;*

(ii) $X_t \xrightarrow{d} X_\infty$ *and* $\mathbb{E}[X_t^{\lambda_o}] \to \mathbb{E}[X_\infty^{\lambda_o}] < \infty$;

(iii) $\mathbb{E}[X_t^\lambda] \to \mathbb{E}[X_\infty^\lambda]] < \infty, \ \forall \lambda \in [0, \lambda_o]$.

2) Let $\lambda_1 \in (0, \lambda_o)$ *and assume that* $\lim_{t\to\infty} \mathbb{E}[X_t^\lambda] = f(\lambda), \ \lambda \in [\lambda_1, \lambda_o]$. *Then (iii) holds and* f *is well defined on* $[0, \lambda_o]$ *by* $f(\lambda) = \mathbb{E}[X_\infty^\lambda]$.

Corollary (Simplification)

Let $(U_t)_{t\in\mathbb{T}}$, $(V_t)_{t\in\mathbb{T}}$ *and* $(W_t)_{t\in\mathbb{T}}$ *s.t.* U_t *and* V_t *are independent,*

$$W_t \stackrel{d}{=} U_t\, V_t, \quad W_t \xrightarrow{d} U_\infty \quad \text{and} \quad V_t \xrightarrow{d} V_\infty$$

and there exists $\lambda_o > 0$ *such that* $(W_t)_{t\in\mathbb{T}}$ *is* $\lambda_o - UI$. *Then,*

$$U_t \xrightarrow{d} U_\infty \quad \text{and} \quad \mathbb{E}[U_\infty^\lambda] = \frac{\mathbb{E}[W_\infty^\lambda]}{\mathbb{E}[V_\infty^\lambda]}, \ \forall \lambda \in [0, \lambda_o].$$

t-monotony and characterization

For $a, b > 0$,, $\mathfrak{b}_{a,b}$ and $\mathfrak{g}_a$ denote r.v.'s with respectively beta and Gamma distribution . It is well known that

$$\mathfrak{g}_a \overset{d}{=} \mathfrak{b}_{a,b}\, \mathfrak{g}_{a+b} \quad \text{and} \quad \mathfrak{b}_{a,b+c} \overset{d}{=} \mathfrak{b}_{a,b}\, \mathfrak{b}_{a+b,c}, \quad \text{for all } a, b, c > 0, .$$

Definition

Let $t \in (0, \infty)$. A function $f : (0, \infty) \to [0, \infty)$ is t-monotone if

$$f(x) = c + \int_{(0,\infty)} (u - x)_+^{t-1}\, \nu(du), \quad c \geq 0,\ x > 0.$$

Note that : f is t-monotone $\Longrightarrow$ s-monotone $\forall s \in (0, t)$.

Proposition

Let f be t-monotone density function of a r.v. $Z > 0$ $(c = 0)$.

1) $\nu(0, \infty) < \infty$ IFF $Z \overset{d}{=} \mathfrak{b}_t\, X_{(t)}$.

2) $\looparrowright$ Alternative proof for Bernstein's characterization of $\mathcal{CM}$.

t-monotony and stationary excess operator

The continuous-time stationary excess operator is given by

$$\mathcal{E}_t(X) \stackrel{d}{:=} \mathfrak{b}_t \, X_{(t)}, \quad \text{with } \mathfrak{b}_t \text{ independent from } X_{(t)}. \tag{4}$$

$(\mathcal{E}_t)_{t\in\mathbb{T}}$ forms a semigroup of commuting operators. We have

$$\mathbb{P}(\mathcal{E}_1(X) \leq x) = \frac{1}{\mathbb{E}[X]} \int_0^x \mathbb{P}(X > u)\, du, \; x \geq 0, \quad \text{and} \quad \mathcal{E}_{n+1} = \mathcal{E}_1 \circ \mathcal{E}_n.$$

Harkness and Shantaram (1969) solved the discrete time problem of :
1- finding a deterministic normalization speed $c_n, \; n \in \mathbb{N}$, and sufficient conditions such that

$$Z_n := \frac{\mathcal{E}_n(X)}{c_n} \xrightarrow{d} Z_\infty \quad \text{as } n \to \infty. \tag{5}$$

2- describing the set of possible distributions for Z_∞.

Harkness and Shantaram Improved

Natural question : What kind of additional information we can recover from the continuous time problem ? i.e. What are the NCS for

$$Z_t \overset{d}{=} \frac{1}{c_t} \mathfrak{b}_t X_{(t)} \overset{d}{\longrightarrow} Z_\infty \quad \text{when } t \in \mathbb{T} \text{ and } t \to \infty?$$

Direction : $t\mathfrak{b}_t \overset{d}{\longrightarrow} \mathfrak{e}$, where $\mathfrak{e}$ is exponentially distributed and choose

$$(U_t, V_t, W_t) = (t\,\mathfrak{b}_t, \frac{X_{(t)}}{\rho_t}, Z_t), \quad \text{with } \rho_t = t\,c_t.$$

and assume $\mathbb{E}[Z_\infty^{\lambda_o}] < \infty$ for some $\lambda_o \in \mathbb{T} \setminus \{0\}$.

Harkness and Shantaram's problem reformulated : Find NSC on ρ_t, s.t.

$$X_t := \frac{X_{(t)}}{\rho_t} \overset{d}{\longrightarrow} X_\infty \ !$$

Necessarily : $Z_\infty \overset{d}{=} \mathfrak{e}\, X_\infty$, where $\mathfrak{e}$ is exponentially distributed, independent from X_∞.

Main result

Theorem (The Normal Limit Theorem)

1) The following statements are equivalent :

(i) $Z_n \xrightarrow{d} Z_\infty$ *and* $\mathbb{E}[Z_\infty^{\lambda_0}] < \infty$ *for some* $\lambda_0 > 0$;

(ii) $Z_t \xrightarrow{d} Z_\infty$ *and* $\mathbb{E}[Z_\infty^{\lambda_0}] < \infty$ *for some* $\lambda_0 > 0$;

(iii) $X_n \xrightarrow{d} X_\infty$ *and* $\mathbb{E}[X_\infty^{\lambda_0}] < \infty$ *for some* $\lambda_0 > 0$;

(iv) $X_t \xrightarrow{d} X_\infty$ *and* $\mathbb{E}[X_\infty^{\lambda_0}] < \infty$ *for some* $\lambda_0 > 0$;

(v) $\mathbb{E}[X_t^\lambda] \to \mathbb{E}[X_\infty^\lambda]$, *for all* $\lambda \in [0, \infty)$.

(vi) $X_t \xrightarrow{d} X_\infty$ *and* $\limsup_{t\to\infty} \frac{\rho_{t+s}}{\rho_t} < \infty$ *for some* $s \in \mathbb{T} \setminus \{0\}$.

2) Necessarily,

$$\rho_t \overset{+\infty}{\sim} \mathbb{E}[X_\infty]\,\mathbb{E}[X^{t+1}]/\,\mathbb{E}[X^t] \text{ and } \lim_{t\to\infty} \frac{\rho_{t+s}}{\rho_t} = e^{cs}, \quad \forall s \in \mathbb{T}$$

$$Z_\infty \overset{d}{=} \mathfrak{e} X_\infty \quad \text{where } \mathfrak{e} \text{ and } X_\infty \text{ are independent}$$

$$\overset{d}{=} e^{-cs}\, \mathfrak{b}_s\, (Z_\infty)_{(s)}, \quad \forall s \geq 0$$

$$\log X_\infty \overset{d}{=} \textit{Normal}(\log \mathbb{E}[X_\infty] - \frac{c}{2}, \frac{c}{2}).$$

Let $g : (0, \infty) \to \mathbb{R}$, recall given by $\Delta_a g(x) := g(x+a) - g(x)$.
g is said monotone of order $k \in \mathbb{N}$, if

$$(-1)^k \Delta_{a_1} \Delta_{a_2} \cdots \Delta_{a_k} g \le 0, \quad \forall\, a_1, a_2, \cdots a_k > 0,\ k \in \mathbb{N} \setminus \{0\}.$$

- $(-1)^k g^{(k)} \ge 0$ implies that g is monotone of order k;
- Choose $\alpha = \mathbb{E}[X_\infty]$, and define for $t \ge 0$

$$g_X(t) = \log \mathbb{E}[X^t] \quad \text{and} \quad \rho_t = \alpha\ \mathbb{E}[X_{(t)}] = \alpha \frac{\mathbb{E}[X^{t+1}]}{\mathbb{E}[X^t]} = \alpha \exp \Delta_1 g_X(t);$$

- We already know that that g_X is convex, that $t \mapsto \rho_t \nearrow$ and then

$$\frac{\rho_{t+s}}{\rho_t} = \frac{\mathbb{E}[X_{(t+s)}]}{\mathbb{E}[X_{(t)}]} = \exp \Delta_1 \Delta_s g_X(t) \le 1;$$

- If g_X is monotone of order 3 (g'_X is concave), then $t \mapsto \rho_{t+s}/\rho_t \nearrow$;
- $\lim_{t\to\infty} \rho_{t+s}/\rho_t = e^{cs}$ for some $c \ge 0$.

Example

If g_X' is a concave function, then X satisfies the last Theorem. For instance, $\log X$ is an infinite divisible random variable such that its Lévy exponent $g_X = \log \mathcal{M}_X$ has the form

$$g_X(\lambda) = d\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{(0,\infty)} (e^{-\lambda x} - 1 + \lambda x \mathbb{1}_{x\leq 1})\, \pi(dx), \quad \lambda \geq 0,$$

with $d \in \mathbb{R}$, $\sigma \geq 0$. Then g_X' is concave. We have

$$\Delta_1 \Delta_s g(t) = \sigma^2 s + \int_{(0,\infty)} e^{-tx}(1-e^{-x})(1-e^{-sx})\, \pi(dx), \quad t,\, s > 0,$$

and $\lim_{t\to\infty} \rho_{t+s}/\rho_t = e^{\sigma^2 s}$

Merci pour votre attention !

Some References

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